ORIGINAL PAPER

INTEGRAL INEQUALITIES

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Abstract: *Several integral forms of Bergstrom's, Bohr's, Jensen's inequalities will be given below, and other inequalities will be also investigated in this paper.*

Keywords: *Bergstrom's inequality, Bohr's inequality, Jensen's inequality.*

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1. INTRODUCTION

We recall Bergstrom's inequality for complex numbers, see [9].

If $a_1, a_2, ..., a_n \in (0, \infty)$ and $z_1, z_2, ..., z_n \in \mathbf{C}$ we have,

$$\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} + ... + \frac{|z_n|^2}{a_n} \geq \frac{|z_1 + z_2 + ... + z_n|^2}{a_1 + a_2 + ... + a_n} \tag{1}$$

Further on, we also recall another result from [9], useful for our goals.

If $n \in \mathbf{N}$, $n \geq 2$, $z_1, z_2, ..., z_n \in \mathbf{C}$ are complex numbers and $a_1, a_2, ..., a_n \in \mathbf{R} \setminus \{0\}$ with $\sum_{k=1}^{n} a_k \neq 0$, then we have

$$\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} + ... + \frac{|z_n|^2}{a_n} - \frac{|z_1 + z_2 + ... + z_n|^2}{a_1 + a_2 + ... + a_n} = \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n} \frac{|a_i z_j - a_j z_i|^2}{a_i a_j}.$$

Taking into account Consequence 2, see [4], which states that for every $z_1, z_2, ..., z_n$, ($n \geq 2$), a sequence of complex numbers that satisfies $a_1, a_2, ..., a_n \in \mathbf{R} \setminus \{0\}$ with $\sum_{k=1}^{n} a_k > 0$, we have

$$\sum_{1 \leq i < j \leq n} \frac{|a_i z_j - a_j z_i|^2}{a_i a_j} \geq \sum_{k=1}^{n} a_k \sum_{k=1}^{n} \frac{|z_k|^2}{a_k} - \frac{n}{2} \sum_{k=1}^{n} |z_k|^2 - \frac{n}{2} \left| \sum_{k=1}^{n} z_k^2 \right|,$$

and then using these two relations we obtain:

$$\left| \sum_{k=1}^{n} z_k \right|^2 \leq \frac{n}{2} \left[\sum_{k=1}^{n} |z_k|^2 + \left| \sum_{k=1}^{n} z_k^2 \right| \right]. \tag{2}$$

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2. THE RESULTS

The integral variant of discrete inequality of Bergstrom, see (1) will be obtained here, using an elementary method and the definition of the integral.

Theorem 1. *Let a, b*$\in$***R****, a<b, and f, g, h : [a, b]*$\to$ ***R*** *three integrable functions on [a, b], and* $g(x) > 0$ *for any* $x \in [a,b]$*. Then the integrable function* $w:[a,b] \to$ ***C****,* $w(x) = f(x) + ih(x)$ *satisfies the inequality:*

$$\left|\int_a^b w(x)dx\right|^2 \le \int_a^b g(x)dx \int_a^b \frac{|w(x)|^2}{g(x)}dx. \tag{3}$$

Proof: Let n$\in$**N** and $x_k = a + k\frac{b-a}{n}, k \in \{0,1,...,n\}$. By (1), we get the following inequality:

$$\sum_{k=1}^{n}\left|\frac{f(x_k)+ih(x_k)}{\sqrt{g(x_k)}}\right|^2 \ge \left|\frac{\sum_{k=1}^{n} f(x_k) + i\sum_{k=1}^{n} h(x_k)}{\sqrt{\sum_{k=1}^{n} g(x_k)}}\right|^2,$$

or

$$\frac{b-a}{n}\sum_{k=1}^{n}\left[\frac{f^2(x_k)+h^2(x_k)}{g(x_k)}\right] \ge \frac{\left(\frac{b-a}{n}\right)^2\left[\left(\sum_{k=1}^{n} f(x_k)\right)^2 + \left(\sum_{k=1}^{n} h(x_k)\right)^2\right]}{\frac{b-a}{n}\sum_{k=1}^{n} g(x_k)}$$

i.e.

$$\sigma\left(\frac{f^2+h^2}{g}, \Delta_n, x_k\right) \ge \frac{\sigma^2(f,\Delta_n,x_k)+\sigma^2(h,\Delta_n,x_k)}{\sigma(g,\Delta_n,x_k)}$$

where $\sigma(\frac{f^2+h^2}{g}, \Delta_n, x_k)$ is the corresponding Riemann sum of function $\frac{f^2+h^2}{g}$ of division $\Delta_n = (x_0, x_1, ..., x_n)$ and the intermediate x_k points. When n tends to infinity we have,

$$\int_a^b \frac{|w(x)|^2}{g(x)}dx = \int_a^b \frac{f^2(x)+h^2(x)}{g(x)}dx \ge \frac{\left(\int_a^b f(x)dx\right)^2 + \left(\int_a^b h(x)dx\right)^2}{\int_a^b g(x)dx} = \frac{\left|\int_a^b w(x)dx\right|^2}{\int_a^b g(x)dx}.$$

Remark 1.

(i) Using inequality (3) we can also obtain inequality (1).

(ii) We can also think to use inequality (2.21), see [2], for m=p=1, first for |f| and g and second for |h| and g and summing we obtain the desired inequality.

The integral variant of the discrete inequality (2) will be obtained below using definition of the integral. It can be also obtained from Corollary 1, see [8].

Remark 2. *If f, h : [a, b]→ **R** are integrable functions on [a, b], where a, b∈ **R**, a< b and* $w:[a,b]\to \mathbf{C}$, $w(x) = f(x) + ih(x)$ *then,*

$$\left|\int_a^b w(x)dx\right|^2 \le \frac{b-a}{2}\left(\int_a^b |w(x)|^2 dx + \left|\int_a^b w^2(x)dx\right|\right).$$

Proof: Let n∈**N** and $x_k = a + k\frac{b-a}{n}, k \in \{0,1,...,n\}$. By (2), we have

$$\left(\sum_{k=1}^n f(x_k)\right)^2 + \left(\sum_{k=1}^n h(x_k)\right)^2 \le$$

$$\le \frac{n}{2}\left\{\sum_{k=1}^n (f^2(x_k) + h^2(x_k)) + \left[\left(\sum_{k=1}^n (f^2(x_k) - h^2(x_k))\right)^2 + \left(\sum_{k=1}^n 2f(x_k)h(x_k)\right)^2\right]^{\frac{1}{2}}\right\},$$

or

$$\left(\sum_{k=1}^n \frac{b-a}{n} f(x_k)\right)^2 + \left(\sum_{k=1}^n \frac{b-a}{n} h(x_k)\right)^2 \le$$

$$\le \frac{b-a}{2}\left\{\frac{b-a}{n}\sum_{k=1}^n (f^2(x_k) + h^2(x_k)) + \frac{b-a}{n}\left[\left(\sum_{k=1}^n (f^2(x_k) - h^2(x_k))\right)^2 + \left(\sum_{k=1}^n 2f(x_k)h(x_k)\right)^2\right]^{\frac{1}{2}}\right\}.$$

When n tends to infinity we obtain,

$$\left(\int_a^b f(x)dx\right)^2 + \left(\int_a^b h(x)dx\right)^2 \le$$

$$\le \frac{b-a}{2}\{\int_a^b [f^2(x) + h^2(x)]dx + [(\int_a^b (f^2(x) - h^2(x))dx)^2 + (\int_a^b 2f(x)h(x)dx)^2]^{\frac{1}{2}}\}$$

i.e. the desired inequality.

It is known, see [5], that

$$\max_{1\le i\le n}\{\frac{p_i}{q_i}\}J_n(f,x,q) \ge J_n(f,x,p) \ge \min_{1\le i\le n}\{\frac{p_i}{q_i}\}J_n(f,x,q)(\ge 0), \quad (4)$$

where

$$J_n(f,x,p) = \sum_{i=1}^n p_i f(x_i) - f(\sum_{i=1}^n p_i x_i),$$

$\sum_{i=1}^n p_i = 1, p_i \ge 0, \sum_{i=1}^n q_i = 1, q_i > 0, x_i \in \mathbf{C}$ and f : **C**→ **R** is convex.

Theorem 2. *Let a, b∈**R**, a<b, f : **R**→ **R** a convex and continuous function and h, l :[a, b]→ **R** two positive and integrable functions on [a, b]. If g : [a, b] → **R** is integrable on [a, b] then*

$$\sup_{x\in[a,b]}\{\frac{h(x)}{l(x)}\}[\int_a^b l(x)f(g(x))dx - f\left(\frac{\int_a^b l(x)g(x)dx}{\int_a^b l(x)dx}\right)\int_a^b l(x)dx] \geq$$

$$\geq \int_a^b h(x)f(g(x))dx - f\left(\frac{\int_a^b h(x)g(x)dx}{\int_a^b h(x)dx}\right)\int_a^b h(x)dx \geq \tag{5}$$

$$\geq \inf_{x\in[a,b]}\{\frac{h(x)}{l(x)}\}[\int_a^b l(x)f(g(x))dx - f\left(\frac{\int_a^b l(x)g(x)dx}{\int_a^b l(x)dx}\right)\int_a^b l(x)dx].$$

Proof: If we consider instead of $p_i, i\in\{1,...,n\}$, $\frac{a_i}{\sum_{i=1}^n a_i}$ with $a_i>0$, $i\in\{1,...,n\}$ and instead of q_i, $\frac{b_i}{\sum_{i=1}^n b_i}, b_i>0$, $i\in\{1,...,n\}$ in inequality (4) then we will have

$$\max_{1\le i\le n}\left\{\frac{a_i}{b_i}\frac{\sum_{i=1}^n b_i}{\sum_{i=1}^n a_i}\right\}\left\{\sum_{i=1}^n \frac{b_i f(x_i)}{\sum_{i=1}^n b_i} - f\left(\sum_{i=1}^n \frac{b_i x_i}{\sum_{i=1}^n b_i}\right)\right\} \geq \sum_{i=1}^n \frac{a_i f(x_i)}{\sum_{i=1}^n a_i} - f\left(\sum_{i=1}^n \frac{a_i x_i}{\sum_{i=1}^n a_i}\right) \geq$$

$$\geq \min_{1\le i\le n} x\left\{\frac{a_i}{b_i}\frac{\sum_{i=1}^n b_i}{\sum_{i=1}^n a_i}\right\}\left\{\sum_{i=1}^n \frac{b_i f(x_i)}{\sum_{i=1}^n b_i} - f\left(\sum_{i=1}^n \frac{b_i x_i}{\sum_{i=1}^n b_i}\right)\right\}.$$

Let n∈**N** and $x_k = a + k\frac{b-a}{n}, k\in\{0,1,...,n\}$. Using previous inequality we get the following:

$$\frac{\sum_{i=1}^n l(x_i)}{\sum_{i=1}^n h(x_i)}\max_{1\le i\le n}\left\{\frac{h(x_i)}{l(x_i)}\right\}\left\{\sum_{i=1}^n \frac{l(x_i)f(g(x_i))}{\sum_{i=1}^n l(x_i)} - f\left(\sum_{i=1}^n \frac{l(x_i)g(x_i)}{\sum_{i=1}^n l(x_i)}\right)\right\} \geq$$

$$\geq \sum_{i=1}^n \frac{h(x_i)f(g(x_i))}{\sum_{i=1}^n h(x_i)} - f\left(\sum_{i=1}^n \frac{h(x_i)g(x_i)}{\sum_{i=1}^n h(x_i)}\right) \geq$$

$$\geq \frac{\sum_{i=1}^n l(x_i)}{\sum_{i=1}^n h(x_i)}\min_{1\le i\le n}\left\{\frac{h(x_i)}{l(x_i)}\right\}\left\{\sum_{i=1}^n \frac{l(x_i)f(g(x_i))}{\sum_{i=1}^n l(x_i)} - f\left(\sum_{i=1}^n \frac{l(x_i)g(x_i)}{\sum_{i=1}^n l(x_i)}\right)\right\}.$$

As in the proof of Theorem 1, when n tends to infinity we have

$$\sup_{x\in[a,b]}\left\{\frac{h(x)}{l(x)}\right\}\frac{\int_a^b l(x)dx}{\int_a^b h(x)dx}\left[\frac{\int_a^b l(x)f(g(x))dx}{\int_a^b l(x)dx}-f\left(\frac{\int_a^b l(x)g(x)dx}{\int_a^b l(x)dx}\right)\right]\geq$$

$$\geq\frac{\int_a^b h(x)f(g(x))dx}{\int_a^b h(x)dx}-f\left(\frac{\int_a^b h(x)g(x)dx}{\int_a^b h(x)dx}\right)\geq$$

$$\geq\inf_{x\in[a,b]}\left\{\frac{h(x)}{l(x)}\right\}\frac{\int_a^b l(x)dx}{\int_a^b h(x)dx}\left[\frac{\int_a^b l(x)f(g(x))dx}{\int_a^b l(x)dx}-f\left(\frac{\int_a^b l(x)g(x)dx}{\int_a^b l(x)dx}\right)\right],$$

taking into account that f is continuous.

Remark 3. *If* $g(x)=x$ *then the above inequality becomes*

$$\sup_{x\in[a,b]}\{\frac{h(x)}{l(x)}\}[\int_a^b l(x)f(x)dx-f\left(\frac{\int_a^b xl(x)dx}{\int_a^b l(x)dx}\right)\int_a^b l(x)dx]\geq$$

$$\geq\int_a^b h(x)f(x)dx-f\left(\frac{\int_a^b xh(x)dx}{\int_a^b h(x)dx}\right)\int_a^b h(x)dx\geq$$

$$\geq\inf_{x\in[a,b]}\{\frac{h(x)}{l(x)}\}[\int_a^b l(x)f(x)dx-f\left(\frac{\int_a^b xl(x)dx}{\int_a^b l(x)dx}\right)\int_a^b l(x)dx].$$

Theorem 3. *Let* $f:\mathbf{R}\to\mathbf{R}$ *a convex and continuous function,* $g:[a,b]\to\mathbf{R}$ *an integrable function on* $[a,b]$, p, $q_1:[a,b]\to(0,\infty)$ *two integrable functions on* $[a,b]$ *and* $q\in(0,1)$. *If there exist*

$M\geq m\geq 0$ *such that* $Mp(x)\geq q_1(x)\geq mp(x),(\forall)x\in\mathbf{R}$ *then:*

$$M^{\frac{1}{q}}\left(\frac{\int_a^b p(x)dx}{\int_a^b q_1(x)dx}\right)^{\frac{1}{q}}\left[\frac{\int_a^b p(x)f(g(x))dx}{\int_a^b p(x)dx}-f\left(\frac{\int_a^b p(x)g(x)dx}{\int_a^b p(x)dx}\right)\right]\geq$$

$$\geq\frac{\int_a^b q_1(x)f(g(x))dx}{\int_a^b q_1(x)dx}-f\left(\frac{\int_a^b q_1(x)g(x)dx}{\int_a^b q_1(x)dx}\right)\geq$$

$$\geq m^{\frac{1}{q}}\left(\frac{\int_a^b p(x)dx}{\int_a^b q_1(x)dx}\right)^{\frac{1}{q}}\left[\frac{\int_a^b p(x)f(g(x))dx}{\int_a^b p(x)dx}-f\left(\frac{\int_a^b p(x)g(x)dx}{\int_a^b p(x)dx}\right)\right].$$

Proof: We will use Proposition 2 from [6] and the proof of Theorem 2.

The following three inequalities are the integral variant of three discrete inequalities of Bohr type from [3].

Proposition 1. *Let* $\alpha, \beta : [a,b] \to \mathbf{R}$ *be two integrable functions on [a, b],* $p, q \in \mathbf{R}$ *and* $z_1, z_2 \in \mathbf{C}$ *(or A, B* $\in \mathcal{B}(\mathcal{H})$*).*

If

$$\int_a^b \alpha^2(x)dx \geq p(b-a), \int_a^b \beta^2(x)dx \geq q(b-a),$$

and

$$\left(\int_a^b \alpha^2(x)dx - p(b-a)\right)\left(\int_a^b \beta^2(x)dx - q(b-a)\right) \geq \left(\int_a^b \alpha(x)\beta(x)dx\right)^2$$

then

$$\int_a^b |\alpha(t)z_1 + \beta(t)z_2|^2 \, dt \geq (b-a)[p|z_1|^2 + q|z_2|^2]$$

or

$$\int_a^b |\alpha(t)A + \beta(t)B|^2 \, dt \geq (b-a)[p|A|^2 + q|B|^2].$$

If

$$\int_a^b \alpha^2(x)dx \leq p(b-a), \int_a^b \beta^2(x)dx \leq q(b-a),$$

and

$$\left(\int_a^b \alpha^2(x)dx - p(b-a)\right)\left(\int_a^b \beta^2(x)dx - q(b-a)\right) \geq \left(\int_a^b \alpha(x)\beta(x)dx\right)^2$$

then the reverse inequality is true.

Proof: We will use Theorem 21 from [3].

Proposition 2. *Let* $\alpha, \beta, \lambda, \mu : [a, b] \to \mathbf{R}$ *integrable on [a, b] and* $z_1, z_2 \in \mathbf{C}$ *(or A, B* $\in \mathcal{B}(\mathcal{H})$*).*

If

$$\int_a^b \alpha^2(x)dx \geq \int_a^b \lambda^2(x)dx, \int_a^b \beta^2(x)dx \geq \int_a^b \mu^2(x)dx$$

and

$$\int_a^b \alpha(x)\beta(x)dx = \int_a^b \lambda(x)\mu(x)dx$$

then

$$\int_a^b |\alpha(t)z_1 + \beta(t)z_2|^2 \, dt \geq \int_a^b |\lambda(t)z_1 + \mu(t)z_2|^2 \, dt$$

or

$$\int_a^b |\alpha(t)A+\beta(t)B|^2\ dt \geq \int_a^b |\lambda(t)A+\mu(t)B|^2\ dt.$$

If

$$\int_a^b \alpha^2(x)dx \leq \int_a^b \lambda^2(x)dx, \int_a^b \beta^2(x)dx \leq \int_a^b \mu^2(x)dx$$

and

$$\int_a^b \alpha(x)\beta(x)dx = \int_a^b \lambda(x)\mu(x)dx$$

then the reverse inequality is true.

Proof: We will use the proof of Theorem 23 from [3].

Proposition 3. *Let* α, $p : [a, b] \to \boldsymbol{R}$ *two integrable functions on [a, b] with the properties:*

$\alpha^2(x) \leq p(x), (\forall)x \in \boldsymbol{R}$ *and* $(\alpha^2(x)-p(x))(\alpha^2(y)-p(y)) \geq \alpha^2(x)\alpha^2(y), (\forall)x, y \in \boldsymbol{R}.$

If $z : [a, b] \to \boldsymbol{C}$ *is integrable on [a, b] then*

$$\left|\int_a^b \alpha(x)z(x)dx\right|^2 \leq \int_a^b p(x)\,|z(x)|^2\ dx.$$

Proof: We will use Theorem 26 from [3] for $\operatorname{Re} z(x)$ and $\operatorname{Im} z(x)$ and the fact that $\frac{b-a}{n} < 1.$

Proposition 4. *If the conditions of Theorem 3.1 from [7] are satisfied then*

$$\left|\int_a^b a_1(x)z(x)dx\right|^2 + \left|\int_a^b b_1(x)z(x)dx\right|^2 \leq \int_a^b c(x)\,|z(x)|^2\ dx,$$

where $a_1, b_2, c : [a,b] \to \boldsymbol{R}$ *are integrable on [a, b],* $z : [a, b] \to \boldsymbol{C}$ *is integrable on [a, b] and* $a_1(x)a_1(y)+b_1(x)b_1(y) \leq c(x), (\forall)x, y \in \boldsymbol{R}.$

Proof: We will apply Theorem 3.1 from [7] for $\operatorname{Re} z(x)$ and $\operatorname{Im} z(x)$ and then we will add the two inequalities.

Proposition 5. *If* $\alpha_k, p : [a,b] \to \boldsymbol{R}$, $k \in \{1,...,m\}$ *are integrable on [a, b],* $z : [a, b] \to \boldsymbol{C}$ *is integrable on [a, b],* $\sum_{k=1}^m \alpha_k^2(x) - p(x) \leq 0, (\forall)x \in \boldsymbol{R}$ *and*

$$\left(\sum_{k=1}^m \alpha_k^2(x) - p(x)\right)\left(\sum_{k=1}^m \alpha_k^2(y) - p(y)\right) \geq \left(\sum_{k=1}^m \alpha_k(x)\alpha_k(y)\right)^2, (\forall)x, y \in \mathbf{R},$$

then

$$\sum_{k=1}^m \left|\int_a^b \alpha_k(x)z(x)dx\right|^2 \geq \int_a^b p(x)\,|z(x)|^2\ dx.$$

Proof: We will apply Theorem 28 from [3] for $\operatorname{Re} z(x)$ and $\operatorname{Im} z(x)$.

The integral form of the inequality of Theorem 3 from [9] will be stated below.

Theorem 4. *Let w : [a, b] → **C** an integrable function on [a, b], given by w(x)=f(x)+ih(x) and g : [a, b] → **R**, g(x)≠0, (∀) x∈ **R** (b>a) is integrable on [a, b]. Then*

$$\frac{1}{2}\int_a^b\int_a^b \frac{|g(x)w(y)-g(y)w(x)|^2}{g(x)g(y)}dxdy = \int_a^b g(x)dx\int_a^b \frac{|w(x)|^2}{g(x)}dx - |\int_a^b w(x)dx|^2,$$

Proof: We will use Theorem 3 from [9] and the definition of Riemann integral and double integral.

But obviously, by direct calculus, we obtain:

$$\frac{1}{2}\int_a^b\int_a^b \frac{|g(x)w(y)-g(y)w(x)|^2}{g(x)g(y)}dxdy = \frac{1}{2}\int_a^b\int_a^b \frac{(g(x)f(y)-g(y)f(x))^2}{g(x)g(y)}dxdy +$$

$$+\frac{1}{2}\int_a^b\int_a^b \frac{(g(x)h(y)-g(y)h(x))^2}{g(x)g(y)}dxdy = \frac{1}{2}\int_a^b g(x)dx\int_a^b \frac{(f^2(x)+h^2(x))^2}{g(x)}dx - [(\int_a^b f(x)dx)^2 + (\int_a^b h(x)dx)^2].$$

Taking into account Theorem 3 we have the following inequality which is the integral form of Proposition 3 from [4]:

Proposition 6. *Let g : [a, b] → **R**, g(x) > 0, (∀) x∈ **R** integrable on [a, b](or integrable on [a, b] with* $\int_a^b g(x)dx > 0$*) and w : [a, b] → **C**, w(x)=f(x)+ih(x) with f, h two real functions integrable on [a, b]. Then*

$$\frac{1}{2}\int_a^b\int_a^b \frac{|g(x)w(y)-g(y)w(x)|^2}{g(x)g(y)}dxdy \geq \int_a^b g(x)dx\int_a^b \frac{|w(x)|^2}{g(x)}dx - \frac{b-a}{2}\int_a^b |w(x)|^2\, dx - \frac{b-a}{2}|\int_a^b w^2(x)|.$$

Proof: We use Proposition 3 from [4] and the above theorem.

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