DECAYS ON SHORT TIME INTERVALS OF GLOBAL SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN GENERAL DOMAINS

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Abstract. We show as the main result of the paper that if w is a strong global nonzero solution of homogeneous Navier-Stokes equations in $\Omega \subseteq \mathbf{R}^3$ and $\beta \in [1/2, 1)$, then there exist $C_0 > 1$ and $\delta_0 \in (0, 1)$ such that

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} \leq C_0$$

for all $t \geq 0$ and $\delta \in [0, \delta_0]$, where $|||\cdot|||_\beta = ||A^\beta \cdot|| + ||\cdot||$ is the graph norm. So, measuring w in the graph norm, we exclude fast decays of w on short time intervals. Ω covers the cases of the whole space, the half space, the exterior domain and the unbounded domain with a non-compact boundary. If, moreover, the solution w decays sufficiently quickly in the energetic norm $\|\cdot\|$ for $t \to \infty$, then a stronger result holds, namely, there exists $C_1 > 1$ and $\delta_0 \in (0, 1)$ such that

$$\frac{|||w(t)|||_\beta}{||w(t+\delta)||} \leq C_1.$$

for all $t \geq 0$ and $\delta \in [0, \delta_0]$. The same results hold for the global weak solutions if we consider $t \geq T_0$, where T_0 is a sufficiently large positive number.

Keywords: *Navier-Stokes equations, global solution, asymptotic properties, fast decays*
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1. Introduction

In this paper we study decays on short time intervals of global solutions of the Navier-Stokes equations in domains $\Omega \subseteq \mathbf{R}^3$

$$\frac{\partial w}{\partial t} - \Delta w + w \cdot \nabla w + \nabla p = 0 \quad \text{in } \Omega \times (0, \infty), \tag{1}$$

$$\nabla \cdot w = 0 \quad \text{in } \Omega \times (0, \infty), \tag{2}$$

$$w|_{t=0} = w_0, \tag{3}$$

$$w = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{4}$$

where $w = w(x, t) = (w_1(x, t), w_2(x, t), w_3(x, t))$ and $p = p(x, t)$ denote the unknown velocity vector and the pressure and $w_0 = w_0(x) = (w_{01}(x), w_{02}(x), w_{03}(x))$ is a given initial velocity vector.

In [8] Scarpellini studied fast decays of global strong solutions of (1) - (4) on short time intervals. He supposed that Ω is either a smooth bounded domain in $\mathbf{R}^3$ or the infinite layer $\mathbf{R}^2 \times (-1/2, 1/2)$ and proved that a fixed global strong solution w of (1) - (4) cannot go through stages of arbitrarily large decays. More precisely, there exist $C_0 > 1$ and $\delta_0 \in (0, 1)$ such that $||w(t)|| \leq C_0||w(t + \delta)||$ for every $t \geq 0$ and every $\delta \in [0, \delta_0]$, where $|| \cdot ||$ denotes the energetic norm. This inequality holds even if the norm at the left hand side is replaced with then norm $||A^{1/2} \cdot ||$. The results from [8] were further improved in [11] for the case of a smooth bounded domain (and also for the case of periodic boundary conditions): If w is a global strong solution of (1) - (4) then for any $k, l, m \in N \cup \{0\}$ there exist $C = C_0(k, l, m) > 1$, $t_0 = t_0(k, l, m) \geq 0$ and $\delta_0 \in (0, 1)$ such that

$$\left\|\frac{d^k w}{dt^k}(t)\right\|_{m,2} \leq C_0 \left\|\frac{d^l w}{dt^l}(t + \delta)\right\|, \ \forall t \geq t_0, \ \forall \delta \in [0, \delta_0]. \tag{5}$$

In (5) $\|\cdot\|_{m,2}$ denotes the Sobolev norm and $\frac{d^k w}{dt^k}$ is the k-th time derivative of w.

If $k, l = 0$ and $\delta = 0$ then (5) gives

$$\|w(t)\|_{m,2} \leq C_0 \|w(t)\|, \ \forall t \in [1, \infty) \tag{6}$$

with $C_0 = C_0(m)$. Since $\|w(t)\| \leq C_0 e^{-\lambda_1 t}$, where $\lambda_1 > 0$ is the smallest eigenvalue of the Stokes operator A_2, the inequality (6) implies the exponential decay of w for $t \to \infty$ in any Sobolev norm. Results of this type were presented in some other papers. For instance, in [10], the asymptotic decays of solutions and their time derivatives in the Sobolev norms were studied for $\Omega = \mathbf{R}^3$. In [6] or [7] results on decays of solutions in the norms $\|A_2^\alpha \cdot\|$, $\alpha \in [0, 1]$ and $\|A_r^\alpha \cdot\|_r$, $\alpha \in [0, 1)$, $r \in (2, 3)$ (A_r is the Stokes operator in L^r_σ) were studied for Ω in the class uniformly C^3 and regular and for exterior domains.

The main goal of this paper is to present several results on decays on short time intervals of global strong solutions of the Navier-Stokes equations in a wide class of domains $\Omega \subseteq \mathbf{R}^3$, which covers such cases as the whole space, the half-space, the exterior domains or domains with smooth non-compact boundaries. Further, we will be able to prove a stronger version of our results under the condition that the solution decays sufficiently quickly in the energetic norm for $t \to \infty$. Since it is known that at least for some domains global weak solutions become strong after a finite time, we will present our results also in the language of global weak solutions. In the paper we will use the method developed in [11].

2. Preliminaries and results

In the paper $L^q = L^q(\Omega), q \geq 1$ denotes the Lebegue spaces with the norm $\|\cdot\|_q$. If $q = 2$, we denote $\|\cdot\| = \|\cdot\|_2$ and $(\cdot,\cdot)$ is the inner product in L^2. $W^{s,q} = W^{s,q}(\Omega), s \geq 0, q \geq 2$ are the usual Sobolev spaces with the norm $\|\cdot\|_{s,q}$. $L^2_\sigma = L^2_\sigma(\Omega)$, resp. $W^{1,2}_{0,\sigma} = W^{1,2}_{0,\sigma}(\Omega)$, is defined as the closure of $\{\varphi \in C_0^\infty(\Omega)^3; \nabla \cdot \varphi = 0\}$ in $L^2(\Omega)^3$, resp. $W^{1,2}(\Omega)^3$. P_σ denotes the orthogonal projection of $L^2(\Omega)^3$ onto L^2_σ. $A = A_2$ is the Stokes operator on L^2_σ. A is positive selfadjoint with a dense domain $D(A) \subset L^2_\sigma$. If Ω is a uniform C^2 domain or if $\Omega = \mathbf{R}^3$ then $D(A) = W^{1,2}_{0,\sigma}(\Omega) \cap W^{2,2}(\Omega)^3$ and $Au = -P_\sigma \Delta u$, $\forall u \in D(A)$. Let $A^\alpha, \alpha \geq 0$ denote the fractional powers of A and $e^{-At}, t \geq 0$ be the Stokes semigroup generated by $-A$. If E_λ, $\lambda \geq 0$ is the resolution of identity for A, we have $A^\alpha e^{-At} u = \int_0^\infty \lambda^\alpha e^{-\lambda t} dE_\lambda u$ and $\|A^\alpha e^{-At} u\|^2 = \int_0^\infty \lambda^{2\alpha} e^{-2\lambda t} d\|E_\lambda u\|^2$ for every $\alpha \geq 0$, $t > 0$ and $u \in L^2_\sigma$ and $A^\alpha u = \int_0^\infty \lambda^\alpha dE_\lambda u$ and $\|A^\alpha u\|^2 = \int_0^\infty \lambda^{2\alpha} d\|E_\lambda u\|^2$ for every $\alpha \geq 0$ and $u \in D(A^\alpha)$. $|||\cdot|||_\beta$, $\beta \geq 0$ denotes the graph norm which is defined as $|||\cdot|||_\beta = \|A^\beta \cdot\| + \|\cdot\|$. Finally, $B(w, w) = P_\sigma(w \cdot \nabla w)$.

The results concerning global strong solutions will be presented for the following class of domains Ω:

Assumption 1 *Let $\Omega = \mathbf{R}^3$ or $\Omega \subseteq \mathbf{R}^3$ be of the class uniformly C^3 and regular, that is the boundary $\partial\Omega$ consists of finitely many disjoint simple C^3 curves (see [6], Assumption, or [13], Definition 1.2.2).*

Results concerning the global weak solutions will be proved under the following stronger assumption on Ω.

Assumption 2 *Let one of the following conditions be satisfied:*

(1) *Ω is the whole space $\mathbf{R}^3$;*

(2) *Ω is the half space $\mathbf{R}^3_+$;*

(3) *Ω satisfies Assumption 1 and has a compact boundary.*

Let $w_0 \in L^2_\sigma$. A measurable function w defined on $\Omega \times (0, \infty)$ is called a global weak solution of (1) - (4) if

$$w \in L^\infty((0, \infty); L^2_\sigma) \cap L^2((0, T); W^{1,2}_{0,\sigma}), \ \forall T > 0 \tag{7}$$

and the integral relation

$$\int_0^T [-(w(t), \partial_t \phi(t)) + (\nabla w(t), \nabla \phi(t)) + (w(t) \cdot \nabla w(t), \phi(t))] dt = (w_0, \phi(0))$$

holds for every $T > 0$ and for all $\phi \in C^1([0, T]; W^{1,2}_{0,\sigma})$ such that $\phi(\cdot, T) = 0$. We say that the global weak solution satisfies the strong energy inequality if

$$\|w(t)\|^2 + 2 \int_s^t \|\nabla w(\sigma)\|^2 d\sigma \leq \|w(s)\|^2$$

for $s = 0$ and almost all $s > 0$, and all $t \geq s$. The existence of global weak solutions satisfying the strong energy inequality was proved by several methods (see, for example [14]).

We will also use the following concept of a global strong solution: Let $w_0 \in D(A^{1/4})$. Then the function $w \in C([0,\infty); D(A^{1/4})) \cap C((0,\infty); D(A)) \cap C^1((0,\infty); L^2_\sigma)$ is called a global strong solution of (1) - (4) if $w(0) = w_0$ and $\frac{dw}{dt} + Aw + P_\sigma(w \cdot \nabla w) = 0$ for every $t > 0$. w satisfies the energy equality (see [6], Remark 2)

$$\|w(t)\|^2 + 2\int_0^t \|\nabla w(\sigma)\|^2 d\sigma = \|w_0\|^2, \ \forall t \geq 0.$$

It implies that $w \in L^\infty((0,\infty); L^2_\sigma)$ and there exists an increasing sequence of positive real numbers $\{t_n\}_{n=1}^\infty$ such that $t_n \to \infty$ and $\|A^{1/2}w(t_n)\| = \|\nabla w(t_n)\| \to 0$ if $n \to \infty$. Since also

$$\|A^{1/4}u\| \leq \|A^{1/2}u\|^{1/2}\|u\|^{1/2}, \ \forall u \in D(A^{1/2}), \tag{8}$$

it is possible to derive from [6], Theorem 1 and Corollary 1, that w exhibits the following asymptotic decay:

$$\|A^\alpha w(t)\| = O(t^{-\alpha}), \ t \to \infty, \ \forall \alpha \in [0,1]. \tag{9}$$

Further, it is possible to prove (also by the use of Lemma 3.2 from [6]) that every global strong solution is simultaneously a global weak solution and, since $D(A^{1/4})$ is continuously embedded into L^3 (see [6], Lemma 2.1), every global strong solution belongs to the space $L^\infty((0,\infty); L^3)$.

If Assumption 2 is satisfied then every global weak solution w satisfying the strong energy inequality becomes strong after a finite time: there exists $T_0 = T_0(w) \geq 0$ such that

$$w \in C([T_0,\infty); D(A^{1/4})) \cap C((T_0,\infty); D(A)) \cap C^1((T_0,\infty); L^2_\sigma). \tag{10}$$

It follows immediately from the above mentioned properties of strong global solutions and by the application of [6], Theorem 1 and [5], Theorem 4. We can conclude, that if Assumption 2 is satisfied then every global weak solution w satisfying the strong energy inequality exhibit the asymptotic decay (9).

The following theorems are the main results of the paper.

Theorem 3 *Let Assumption 1 hold, $w_0 \in D(A^{1/4})$ and $w_0 \neq 0$. Let w be a global strong solution of (1) - (4). Let $\beta \in [1/2, 1)$. Then there exist $C_0 > 1$ and $\delta_0 \in (0,1)$ such that*

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} \leq C_0, \ \forall t \geq 1, \ \forall \delta \in (0, \delta_0]. \tag{11}$$

Theorem 4 *Let Assumption 1 hold, $w_0 \in D(A^{1/4})$, $w_0 \neq 0$. Let w be a global strong solution of (1) - (4). Let $\beta \in [1/2, 1)$. Let further*

$$\|w(t)\| \leq Ct^\kappa, \ \forall t \in [1,\infty) \tag{12}$$

for some $\kappa < -1$ and $C > 0$. Then there exist $C_1 > 1$ and $\delta_0 \in (0,1)$ such that

$$\frac{|||w(t)|||_\beta}{||w(t+\delta)||} \leq C_1, \ \forall t \geq 1, \ \forall \delta \in [0, \delta_0]. \tag{13}$$

Corollary 5 *Let Assumption 2 hold and $w_0 \in L^2_\sigma$, $w_0 \neq 0$. Let w be a global weak solution of (1) - (4) satisfying the strong energy inequality and let T_0 is from (10). Let $\beta \in [1/2, 1)$. Then there exist $C_0 > 1$ and $\delta_0 \in (0,1)$ such that*

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} \leq C_0, \ \forall t \geq T_0 + 1, \ \forall \delta \in (0, \delta_0]. \tag{14}$$

Moreover, if

$$\|w(t)\| \leq Ct^\kappa, \ \forall t \in [1,\infty)$$

for some $\kappa < -1$ and $C > 0$, then there exist $C_1 > 1$ and $\delta_0 \in (0,1)$ such that

$$\frac{|||w(t)|||_\beta}{||w(t+\delta)||} \leq C_1, \ \forall t \geq T_0 + 1, \ \forall \delta \in [0, \delta_0]. \tag{15}$$

The following result from [9] shows that the class of solutions satisfying the assumptions of Theorem 4 is not empty: Let $\Omega = \mathbf{R}^3$ and let w be a weak suitable solution in the sense of Cafarelli, Kohn and Nirenberg (see [2]) with zero average initial data outside a class of functions of radially equidistributed energy. Then there exists a constant C_0 depending only on norms of the initial data such that

$$\|w(t)\| \leq C_0(t+1)^{-5/4}. \tag{16}$$

Moreover, this solution satisfies the strong energy inequality, as was proved in the appendix of [15]. So, such solutions satisfy the assumptions of Theorem 4 with $\kappa = -5/4$.

Let us also remark that unlike the case of a bounded domain, we do not have the inequality $\|B(w,w)\| \leq \|A^{1/2}w\| \, ||A^\beta w||$, $\beta \in [3/4, 1)$ in the case of the unbounded domains, which must be replaced by $\|B(w,w)\| \leq \|A^{1/2}w\| \, (||A^\beta w|| + ||w||)$. It leads to the use of the graph norm $|||\cdot|||_\beta = ||A^\beta \cdot|| + ||\cdot||$ in (11) and (14). Theorem 3 says that if we measure a global strong solution w in the graph norm, then fast decays of w on short time intervals are excluded. Theorem 4 strengthens the result from Theorem 3 (the graph norm in the denominator of (11) is replaced by the weaker energetic norm in the denominator of (13)) under the condition that the solution exhibits a sufficiently quick decay in the energetic norm for $t \to \infty$. We do not know, whether or not the inequality (13) holds for every global strong solution. Corollary 5 is a reformulation of results from Theorems 3 and 4 for the case of global weak solutions under the condition that Assumption 2 holds. The inequalities (14) and (15) hold only for $t \geq T_0 + 1$ because of a possible existence of blow-ups in the time interval $[0, T_0]$.

It has been already mentioned in Introduction that much stronger results on decays of global solutions on short time intervals can be achieved if Ω is a smooth bounded domain (see (5) or [11] and realize that the Sobolev norm $\|\cdot\|_{m,2}$ on the left hand side of (5) is equivalent to the norm $\|A^{m/2}\cdot\|$ at least for $m \in (0,2)$). We do not know whether the results for smooth bounded domains can be generalized for the case of unbounded domains.

3. Auxiliary results

Let us present, at first, several auxiliary results. The following inequality can be derived as a consequence of Hölder inequality and Lemma 2.4.3 form [12]: if $\gamma \in [3/4, 1)$ then there exists $c_1 > 0$ such that

$$\|B(u,u)\| \leq c_1 \|A^{1/2}u\| \, |||u|||_\gamma, \ \forall u \in D(A^\gamma). \tag{17}$$

It is possible to prove elementarily that if $0 \leq \beta \leq \alpha$ then

$$|||u|||_\beta \leq 3|||u|||_\alpha, \ \forall u \in D(A^\alpha). \tag{18}$$

If $\alpha \in [0,1)$ then there exists $c_1 > 0$ such that (see [13])

$$\|A^\alpha e^{-At}u\| \leq \frac{c_1}{t^\alpha}\|u\|, \ \forall t > 0, \ \forall u \in L^2_\sigma. \tag{19}$$

The moment inequality (see e.g. [13]) can be proved on the basis of the fact that $A^\alpha u = \int_0^\infty \lambda^\alpha dE_\lambda u$ for every $\alpha \geq 0$: If $0 \leq z < y < x$ and $u \in D(A^x)$ then

$$\|A^y u\| \leq \|A^z u\|^{\frac{x-y}{x-z}} \|A^x u\|^{\frac{y-z}{x-z}}.$$

The following lemma and its corollary are substantial for the proof of Theorems 3 and 4. They say, roughly speaking, that in the Stokes problem the decay of the solution on a time interval is always smaller than the decay on the preceding interval of the same length.

Lemma 6 *If $w \in D(A^\alpha)$, $w \neq 0$, $t \geq 0$ and $0 \leq \beta \leq \alpha$ then*

$$\frac{\|A^\alpha w\|}{\|A^\beta e^{-At}w\|} \geq \frac{\|A^\alpha e^{-At}w\|}{\|A^\beta e^{-2At}w\|}.$$

Proof: Let E_λ, $\lambda \geq 0$ is the resolution of identity for the Stokes operator A. Then

$$\|A^\beta e^{-At}w\|^2 = \int_0^\infty \lambda^{2\beta} e^{-2\lambda t} d\|E_\lambda w\|^2, \quad t \geq 0. \tag{20}$$

By the Hölder inequality we get easily that

$$\|A^{\beta}e^{-At}w\|^2 = \int_0^{\infty} \lambda^{2\beta}e^{-2\lambda t}d\|E_{\lambda}w\|^2 \leq$$
$$\left(\int_0^{\infty} \lambda^{2\beta}d\|E_{\lambda}w\|^2\right)^{1/2}\left(\int_0^{\infty} \lambda^{2\beta}e^{-4\lambda t}d\|E_{\lambda}w\|^2\right)^{1/2} = \|A^{\beta}w\|\|A^{\beta}e^{-2At}w\|$$

and immediately

$$\frac{\|A^{\beta}w\|}{\|A^{\beta}e^{-At}w\|} \geq \frac{\|A^{\beta}e^{-At}w\|}{\|A^{\beta}e^{-2At}w\|}. \tag{21}$$

We will show further that the function $t \mapsto \|A^{\alpha}e^{-At}w\|^2/\|A^{\beta}e^{-At}w\|^2$ is non-increasing. Firstly, for every $\gamma \geq 0$

$$\frac{d}{dt}\|A^{\gamma}e^{-At}w\|^2 = -2\|A^{\gamma+1/2}e^{-At}w\|^2, \quad t > 0$$

and therefore

$$\frac{d}{dt}\frac{\|A^{\alpha}e^{-At}w\|^2}{\|A^{\beta}e^{-At}w\|^2} = \frac{2\|A^{\alpha}e^{-At}w\|^2\|A^{\beta+1/2}e^{-At}w\|^2 - 2\|A^{\alpha+1/2}e^{-At}w\|^2\|A^{\beta}e^{-At}w\|^2}{\|A^{\beta}e^{-At}w\|^4}, \quad t > 0.$$

Further,

$$\|A^{\alpha}e^{-At}w\|^2\|A^{\beta+1/2}e^{-At}w\|^2 \leq \|A^{\alpha+1/2}e^{-At}w\|^2\|A^{\beta}e^{-At}w\|^2,$$

as follows from the moment inequality. So,

$$\frac{d}{dt}\frac{\|A^{\alpha}e^{-At}w\|^2}{\|A^{\beta}e^{-At}w\|^2} \leq 0, \quad t > 0$$

and due to the continuity from the right at 0 we get that the above mentioned function is non-increasing. It means especially, that

$$\frac{\|A^{\alpha}w\|^2}{\|A^{\beta}w\|^2} \geq \frac{\|A^{\alpha}e^{-At}w\|^2}{\|A^{\beta}e^{-At}w\|^2}, \quad t \geq 0. \tag{22}$$

Using now (21) and (22), we get

$$\frac{\|A^{\alpha}w\|}{\|A^{\beta}e^{-At}w\|} = \frac{\|A^{\alpha}w\|}{\|A^{\beta}w\|}\frac{\|A^{\beta}w\|}{\|A^{\beta}e^{-At}w\|} \geq \frac{\|A^{\alpha}e^{-At}w\|}{\|A^{\beta}e^{-At}w\|}\frac{\|A^{\beta}e^{-At}w\|}{\|A^{\beta}e^{-2At}w\|} = \frac{\|A^{\alpha}e^{-At}w\|}{\|A^{\beta}e^{-2At}w\|},$$

which completes the proof of the lemma. ◯

Corollary 7 *If* $w \in D(A^{\alpha})$, $w \neq 0$, $t \geq 0$ *and* $0 \leq \beta \leq \alpha$ *then*

$$\frac{|||w|||_{\alpha}}{|||e^{-At}w|||_{\beta}} \geq \frac{|||e^{-At}w|||_{\alpha}}{|||e^{-2At}w|||_{\beta}}.$$

Proof: The proof of the corollary follows immediately from Lemma 6 and from the elementary fact that if $\frac{\alpha_1}{\beta_1} \geq \frac{\beta_1}{\gamma_1}$ and $\frac{\alpha_2}{\beta_2} \geq \frac{\beta_2}{\gamma_2}$ for some positive $\alpha_i, \beta_i, \gamma_i$, $i = 1, 2$, then $\frac{\alpha_1+\alpha_2}{\beta_1+\beta_2} \geq \frac{\beta_1+\beta_2}{\gamma_1+\gamma_2}$. ◯

4. Decays of solutions on short time intervals

Throughout the following text c denotes the generic constant which can change from line to line.

Proof of Theorem 3: Let the assumptions of Theorem 3 be fulfilled. We will use the method from [11]. We denote

$$H = \max_{t\in[1,\infty)} |||w(t)|||_{\beta}.$$

It follows from (9) that $H < \infty$. Since $t \to \|w(t)\|$ is a continuous function on $[1, \infty)$ and $|||w(t)|||_\beta \geq \|w(t)\| > 0$ for all $t \in [0, \infty)$, there exist $C_0' > 1$ and $\delta_0' \in (0, 1)$ such that

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} \leq C_0', \ \forall t \in [1, 3], \ \forall \delta \in (0, \delta_0']. \tag{23}$$

If $\beta \in [1/2, 3/4)$, we fix $\gamma \in (3/4, 1)$ such that $3/2 - (\beta + \gamma) > 0$. We set now $D_0 = 6C_0'$ and let $\delta_0 \in (0, \delta_0']$ be such a number that

$$4Hc\left(D_0 e^{\frac{5D_0}{2(D_0-1)}}\right)^3 \frac{\delta_0^{1-\beta}}{1-\beta} \leq 1, \ \text{if } \beta \in [3/4, 1) \tag{24}$$

and

$$4Hc\left(D_0 e^{\frac{5D_0}{2(D_0-1)}}\right)^3 \delta_0^{3/2-(\beta+\gamma)} \leq 1, \ \text{if } \beta \in [1/2, 3/4). \tag{25}$$

We will prove the following proposition:

Proposition P: Let $t > 3$, $\delta \in (0, \delta_0]$. Let further

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} = C \in \left(D_0, D_0 e^{\frac{5D_0}{2(D_0-1)}}\right) \tag{26}$$

and

$$|||w(t)|||_\beta \geq |||w(s)|||_\beta, \ \forall s \in [t, t+\delta]. \tag{27}$$

Then there exists $t^* \in [t-\delta, t)$ such that

$$\frac{|||w(t^*)|||_\beta}{|||w(t)|||_\beta} \geq \frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} \frac{\left(1 - \frac{|||w(t)|||_\beta}{2H}\right)^2}{\left(1 + \frac{|||w(t)|||_\beta}{2H}\right)}. \tag{28}$$

So let the assumptions of Proposition P be fulfilled. We can suppose that

$$\max_{s \in [t-\delta, t]} |||w(s)|||_\beta < C|||w(t)|||_\beta, \tag{29}$$

because otherwise (28) would be satisfied immediately. We begin with the integral representation of w:

$$w(t+\delta) = e^{-A\delta} w(t) + \int_0^\delta e^{-A(\delta-s)} B(w(t+s), w(t+s)) \, ds, \tag{30}$$

$$w(t) = e^{-A\delta} w(t-\delta) + \int_0^\delta e^{-A(\delta-s)} B(w(t-\delta+s), w(t-\delta+s)) \, ds. \tag{31}$$

Suppose at first that $\beta \in [3/4, 1)$. Applying gradually (19), (17), (18) and (29) we obtain from (31) that

$$\begin{aligned}
&|||w(t) - e^{-A\delta} w(t-\delta)|||_\beta \leq \\
&\int_0^\delta c(\delta-s)^{-\beta} \, \|B(w(t-\delta+s), w(t-\delta+s)\| \, ds \leq \\
&\int_0^\delta c(\delta-s)^{-\beta} \, \|A^{1/2} w(t-\delta+s)\| \; |||w(t-\delta+s)|||_\beta \, ds \leq \\
&|||w(t)|||_\beta \int_0^\delta c(\delta-s)^{-\beta} \, \frac{|||w(t-\delta+s)|||_{1/2}}{|||w(t-\delta+s)|||_\beta} \frac{|||w(t-\delta+s)|||_\beta}{|||w(t)|||_\beta} \, |||w(t-\delta+s)|||_\beta \, ds \\
&\leq |||w(t)|||_\beta^2 c C^2 \int_0^\delta (\delta-s)^{-\beta} \, ds = |||w(t)|||_\beta^2 c C^2 \frac{\delta_0^{1-\beta}}{1-\beta}.
\end{aligned}$$

So we can get from (24) and (26) that

$$|||w(t) - e^{-A\delta} w(t-\delta)|||_\beta \leq |||w(t)|||_\beta \left(2HcC^2 \frac{\delta_0^{1-\beta}}{1-\beta}\right) \frac{|||w(t)|||_\beta}{2H} \leq |||w(t)|||_\beta \, \frac{|||w(t)|||_\beta}{2H} \tag{32}$$

or

$$|||w(t) - e^{-A\delta}w(t-\delta)|||_\beta \le |||w(t+\delta)|||_\beta \left(4HcC^3 \frac{\delta_0^{1-\beta}}{1-\beta}\right) \frac{|||w(t)|||_\beta}{4H} \le$$

$$|||w(t+\delta)|||_\beta \frac{|||w(t)|||_\beta}{4H}. \tag{33}$$

(33) now gives immediately that

$$|||e^{-A\delta}w(t) - e^{-2A\delta}w(t-\delta)|||_\beta \le |||w(t+\delta)|||_\beta \frac{|||w(t)|||_\beta}{4H}. \tag{34}$$

It follows from (30), (19), (17), (18), (26), (27) and (24) that

$$|||w(t+\delta) - e^{-A\delta}w(t)|||_\beta \le$$

$$\int_0^\delta c(\delta-s)^{-\beta}||A^{1/2}w(t+s)||\;|||w(t+s)|||_\beta\, ds \le$$

$$|||w(t+\delta)|||_\beta \int_0^\delta c(\delta-s)^{-\beta} \frac{|||w(t+s)|||_{1/2}}{|||w(t+s)|||_\beta} \frac{|||w(t+s)|||_\beta}{|||w(t+\delta)|||_\beta}|||w(t+s)|||_\beta\, ds \le$$

$$|||w(t+\delta)|||_\beta\;|||w(t)|||_\beta\; c\,C \int_0^\delta (\delta-s)^{-\beta}\, ds =$$

$$|||w(t+\delta)|||_\beta \left(4HcC\frac{\delta_0^{1-\beta}}{1-\beta}\right) \frac{|||w(t)|||_\beta}{4H} \le |||w(t+\delta)|||_\beta \frac{|||w(t)|||_\beta}{4H}. \tag{35}$$

(34) and (35) provide the estimate

$$||||e^{-2A\delta}w(t-\delta) - w(t+\delta)|||_\beta \le |||e^{-2A\delta}w(t-\delta) - e^{-A\delta}w(t)|||_\beta +$$

$$|||e^{-A\delta}w(t) - w(t+\delta)|||_\beta \le |||w(t+\delta)|||_\beta \frac{|||w(t)|||_\beta}{2H}. \tag{36}$$

It follows now from Corollary 7 and (32) and (36) that

$$|||w(t-\delta)|||_\beta \ge \frac{|||e^{-A\delta}w(t-\delta)|||_\beta^2}{|||e^{-2A\delta}w(t-\delta)|||_\beta} \ge \frac{|||w(t)|||_\beta^2 \left(1 - \frac{|||w(t)|||_\beta}{2H}\right)^2}{|||w(t+\delta)|||_\beta \left(1 + \frac{|||w(t)|||_\beta}{2H}\right)}.$$

So we put $t^* = t - \delta$ and (28) is proved.

Now we suppose that $\beta \in [1/2, 3/4)$. If we use (30), (19) and (17), we get for every $\tau \in (0, \delta]$:

$$|||w(t+\tau)|||_\gamma \le |||e^{-A\tau}w(t)|||_\gamma +$$

$$\|\int_0^\tau A^\gamma e^{-A(\tau-s)}B(w(t+s), w(t+s))\, ds\| + \|\int_0^\tau e^{-A(\tau-s)}B(w(t+s), w(t+s))\, ds\| \le$$

$$\|A^{\gamma-1/2}e^{-A\tau}A^{1/2}w(t)\| + \|w(t)\| + \int_0^\tau c(\tau-s)^{-\gamma}\|B(w(t+s), w(t+s))\|\, ds \le$$

$$\frac{c}{\tau^{\gamma-1/2}}|||w(t)|||_{1/2} + \int_0^\tau c(\tau-s)^{-\gamma}\, \|A^{1/2}w(t+s)\|\;|||w(t+s)|||_\gamma\, ds \le$$

$$\frac{c}{\tau^{\gamma-1/2}} \sup_{\xi\in[0,\delta]} |||w(t+\xi)|||_{1/2} + c \sup_{\xi\in[0,\delta]} |||w(t+\xi)|||_{1/2} \int_0^\tau (\tau-s)^{-\gamma}\; |||w(t+s)|||_\gamma\, ds.$$

Applying now the singular Gronwall inequality (see [1], p.52) we get

$$|||w(t+\tau)|||_\gamma \le \frac{\tilde{c}}{\tau^{\gamma-1/2}} \sup_{\xi\in[0,\delta]} |||w(t+\xi)|||_{1/2},\ \forall\tau \in (0,\delta] \tag{37}$$

and similarly

$$|||w(t-\delta+\tau)|||_\gamma \le \frac{\tilde{c}}{\tau^{\gamma-1/2}} \sup_{\xi\in[0,\delta]} |||w(t-\delta+\xi)|||_{1/2},\ \forall\tau \in (0,\delta], \tag{38}$$

where $\tilde{c}$ is an absolute constant independent of t, which will be included into the generic constant c. We now use (31), (19), (17), (18), (38) and (29) and get

$$|||w(t)-e^{-A\delta}w(t-\delta)|||_\beta \le \int_0^\delta c(\delta-s)^{-\beta}\,\|A^{1/2}w(t-\delta+s)\|\;|||w(t-\delta+s)|||_\gamma\,ds \le$$
$$\int_0^\delta c(\delta-s)^{-\beta}\,|||w(t-\delta+s)|||_\beta \frac{\tilde{c}}{s^{\gamma-1/2}}\sup_{\xi\in[0,\delta]}|||w(t-\delta+\xi)|||_{1/2}\,ds \le$$
$$|||w(t)|||_\beta \int_0^\delta \frac{c}{(\delta-s)^\beta s^{\gamma-1/2}}\frac{|||w(t-\delta+s)|||_\beta}{|||w(t)|||_\beta}\sup_{\xi\in[0,\delta]}|||w(t-\delta+\xi)|||_\beta\,ds \le$$
$$|||w(t)|||_\beta^2 C^2 c\delta_0^{3/2-(\beta+\gamma)}.$$

Using now (25) and (26) we obtain that

$$|||w(t)-e^{-A\delta}w(t-\delta)|||_\beta \le |||w(t)|||_\beta\left(2HcC^2\delta_0^{3/2-(\beta+\gamma)}\right)\frac{|||w(t)|||_\beta}{2H} \le |||w(t)|||_\beta\frac{|||w(t)|||_\beta}{2H} \tag{39}$$

and also

$$|||w(t)-e^{-A\delta}w(t-\delta)|||_\beta \le |||w(t+\delta)|||_\beta\left(4HcC^3\delta_0^{3/2-(\beta+\gamma)}\right)\frac{|||w(t)|||_\beta}{4H} \le$$
$$|||w(t+\delta)|||_\beta\frac{|||w(t)|||_\beta}{4H}. \tag{40}$$

So we have immediately by (40) that

$$|||e^{-A\delta}w(t)-e^{-2A\delta}w(t-\delta)|||_\beta \le |||w(t+\delta)|||_\beta\frac{|||w(t)|||_\beta}{4H}. \tag{41}$$

It follows from (30), (19), (17), (37), (18), (26), (27) and (25) that

$$|||w(t+\delta)-e^{-A\delta}w(t)|||_\beta \le$$
$$\int_0^\delta c(\delta-s)^{-\beta}\,\|A^{1/2}w(t+s)\|\;|||w(t+s)|||_\gamma\,ds \le$$
$$\int_0^\delta c(\delta-s)^{-\beta}\,|||w(t+s)|||_\beta\,\frac{\tilde{c}}{s^{\gamma-1/2}}\sup_{\xi\in[0,\delta]}|||w(t+\xi)|||_{1/2}\,ds \le$$
$$|||w(t+\delta)|||_\beta\int_0^\delta \frac{c}{(\delta-s)^\beta s^{\gamma-1/2}}\frac{|||w(t+s)|||_\beta}{|||w(t+\delta)|||_\beta}\sup_{\xi\in[0,\delta]}|||w(t+\xi)|||_\beta\,ds \le$$
$$|||w(t+\delta)|||_\beta\left(4HcC\delta_0^{3/2-(\beta+\gamma)}\right)\frac{|||w(t)|||_\beta}{4H} \le |||w(t+\delta)|||_\beta\frac{|||w(t)|||_\beta}{4H}. \tag{42}$$

(41) and (42) provide the estimate

$$|||e^{-2A\delta}w(t-\delta)-w(t+\delta)|||_\beta \le |||e^{-2A\delta}w(t-\delta)-e^{-A\delta}w(t)|||_\beta +$$
$$|||e^{-A\delta}w(t)-w(t+\delta)|||_\beta \le |||w(t+\delta)|||_\beta\frac{|||w(t)|||_\beta}{2H}. \tag{43}$$

The proof of (28) is now finished exactly as in the case $\beta\in[3/4,1)$ and the proof of Proposition P is complete.

Let us fix now $t\in[1,\infty)$, $\delta\in(0,\delta_0]$ and suppose that

$$|||w(t)|||_\beta > H/D_0 \text{ and} \tag{44}$$

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} \ge D_0\frac{1+1/2}{(1-1/2)^2} = 6D_0. \tag{45}$$

Since $D_0 > C_0'$ and $\delta_0 \le \delta_0'$, it follows from (23) and (45) that $t>3$. We can also suppose without loss of generality that

$$|||w(t)|||_\beta = \max_{s\in[t,t+\delta]}|||w(s)|||_\beta$$

and (by possible decreasing of δ)

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} = 6D_0.$$

Let us notice that $6D_0 < D_0 e^{\frac{5D_0}{2(D_0-1)}}$ (because $D_0 > 1$) and the conditions (26) and (27) are satisfied. By Proposition P there exists $t^* \in [t-\delta, t)$ so that

$$\frac{|||w(t^*)|||_\beta}{|||w(t)|||_\beta} \geq \frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} \frac{\left(1 - \frac{|||w(t)|||_\beta}{2H}\right)^2}{\left(1 + \frac{|||w(t)|||_\beta}{2H}\right)} \geq 6D_0 \frac{(1-1/2)^2}{1+1/2} = D_0.$$

Thus, by (44), $|||w(t^*)|||_\beta \geq D_0|||w(t)|||_\beta > D_0 H/D_0 = H$ and it is the contradiction with the definition of H. Let $D_1 = 6D_0$. We proved

Proposition P_1**:** Let $t \in [1, \infty)$, $\delta \in (0, \delta_0]$ and $|||w(t)|||_\beta > H/D_0$. Then

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} < D_1.$$

We define now

$$D_n = D_{n-1} \frac{1 + \frac{1}{2D_0 D_1 \ldots D_{n-2}}}{\left(1 - \frac{1}{2D_0 D_1 \ldots D_{n-2}}\right)^2}, \ \forall n \in N, n \geq 2. \tag{46}$$

We have

$$6 < D_0 < D_1 < \ldots < D_{n-1} < D_n, \ \forall n \in N, \tag{47}$$

$$D_n = 6D_0 \prod_{j=0}^{n-2} \frac{1 + \frac{1}{2D_0 D_1 \ldots D_j}}{\left(1 - \frac{1}{2D_0 D_1 \ldots D_j}\right)^2} \leq D_0 \prod_{j=0}^{n-1} \frac{1 + \frac{1}{2D_0^j}}{\left(1 - \frac{1}{2D_0^j}\right)^2}, \ \forall n \geq 2$$

and

$$\ln D_n \leq \ln D_0 + \sum_{j=0}^{n-1} \ln\left(1 + \frac{1}{2D_0^j}\right) - 2\ln\left(1 - \frac{1}{2D_0^j}\right), \ \forall n \geq 1.$$

It follows from the elementary properties of the function $x \to \ln(1+x)$ that

$$\ln D_n < \ln D_0 + \sum_{j=0}^{n-1} \left(\frac{1}{2D_0^j} + 4\frac{1}{2D_0^j}\right) < \ln D_0 + \frac{5D_0}{2(D_0-1)}$$

and

$$D_n < D_0 e^{\frac{5D_0}{2(D_0-1)}}, \ \forall n \in N. \tag{48}$$

We will prove now that for every $n \in N$ the following proposition is valid:

Proposition P_n**:** Let $t \in [1, \infty)$, $\delta \in (0, \delta_0]$ and

$$|||w(t)|||_\beta > \frac{H}{D_0 D_1 \ldots D_{n-1}}.$$

Then

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} < D_n.$$

We will use the mathematical induction. Proposition P_1 has already been proved. Let us suppose that P_n holds for some $n \in N$ and we will prove the validity of P_{n+1}. Thus, let $t \in [1, \infty)$, $\delta \in (0, \delta_0]$ and $|||w(t)|||_\beta > H/D_0 D_1 \ldots D_n$. We can suppose that

$$|||w(t)|||_\beta \leq H/D_0 D_1 \ldots D_{n-1}, \tag{49}$$

since otherwise we would apply Proposition P_n, get $|||w(t)|||_\beta/|||w(t+\delta)|||_\beta < D_n < D_{n+1}$ and Proposition P_{n+1} would be proved. We suppose by contradiction that

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} \geq D_{n+1}. \tag{50}$$

It follows then from (23) and (47) that $t > 3$. We can suppose without loss of generality that

$$|||w(t)|||_\beta \geq |||w(s)|||_\beta,\ \forall s \in [t, t+\delta] \tag{51}$$

and also

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} = D_{n+1}. \tag{52}$$

Due to (47), (48), (51) and (52) we see that (26) and (27) are satisfied. Therefore, Proposition P, (52), (49) and (46) yield that there exists $t^* \in [t-\delta, t)$ so that

$$\frac{|||w(t^*)|||_\beta}{|||w(t)|||_\beta} \geq \frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} \frac{\left(1 - \frac{|||w(t)|||_\beta}{2H}\right)^2}{\left(1 + \frac{|||w(t)|||_\beta}{2H}\right)} \geq D_{n+1} \frac{\left(1 - \frac{1}{2D_0 D_1 \dots D_{n-1}}\right)^2}{\left(1 + \frac{1}{2D_0 D_1 \dots D_{n-1}}\right)} = D_n. \tag{53}$$

If we use the assumptions of Proposition P_{n+1} we obtain that

$$|||w(t^*)|||_\beta \geq D_n |||w(t)|||_\beta > D_n \frac{H}{D_0 D_1 \dots D_n} = \frac{H}{D_0 D_1 \dots D_{n-1}}$$

and according to Proposition P_n we get that

$$\frac{|||w(t^*)|||_\beta}{|||w(t)|||_\beta} < D_n,$$

which is the contradiction to (53). Therefore, (50) does not hold, in fact

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} < D_{n+1}$$

and Proposition P_{n+1} is proved. We proved that Proposition P_n holds for every $n \in N$.

We now finish the proof of Theorem 3. Let us fix $t \in [1, \infty)$ and $\delta \in (0, \delta_0]$. Then there exists $n \in N$ so that $|||w(t)|||_\beta > \frac{H}{D_0 D_1 \dots D_{n-1}}$. By Proposition P_n and by (48) we get that

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} < D_n < D_0 e^{\frac{5D_0}{2(D_0-1)}}.$$

Setting $C_0 = D_0 e^{\frac{5D_0}{2(D_0-1)}}$ the proof of Theorem 3 is complete. ○

Proof of Theorem 4: Let the assumptions of Theorem 4 be fulfilled. We can suppose due to (18) that $\beta \in [3/4, 1)$. Since $w(t) \neq 0$ for all $t \in [1, \infty)$, it follows from the continuity of the function $|||\cdot|||_{1/2}$ on $[1, \infty)$ that there exist $C_0' > 1$ and $\delta_0' \in (0, 1)$ such that

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_{1/2}} \leq C_0',\ \forall t \in [1, 3],\ \forall \delta \in [0, \delta_0']. \tag{54}$$

In the proof the generic constant $c > 0$ will include also the constant C_0 from Theorem 3. Let us now choose $\delta_0 \in (0, \delta_0']$ in such a way that $2\delta_0$ is smaller than δ_0 from Theorem 3 and

$$c\delta_0^{1-\beta}/(1-\beta) \leq 1. \tag{55}$$

Let $\theta = \kappa/4 - 3/4$. We also set

$$D_0 = C_0' e^{-\frac{5}{(\theta+1)\delta_0}} \tag{56}$$

We will show at first that

$$\frac{|||w(t-\delta_0)|||_\beta}{|||w(t)|||_{1/2}} \geq \frac{|||w(t)|||_\beta}{|||w(t+\delta_0)|||_{1/2}} \frac{(1-t^\theta)^2}{(1+t^\theta)},\ \forall t > 3. \tag{57}$$

Thus, let $t > 3$. We begin with the integral representation of w:

$$w(t+\delta_0) = e^{-A\delta_0}w(t) + \int_0^{\delta_0} e^{-A(\delta_0-s)}B(w(t+s), w(t+s))\,ds, \tag{58}$$

$$w(t) = e^{-A\delta_0}w(t-\delta_0) + \int_0^{\delta_0} e^{-A(\delta_0-s)}B(w(t-\delta_0+s), w(t-\delta_0+s))\,ds. \tag{59}$$

Since $\|A^{1/2}w\| \le \|w\|^{1/2}\|Aw\|^{1/2}$, we use (59), (19), (17), Theorem 3, (12), (9) and (55) and get the following estimate:

$$\begin{aligned}
&|||w(t) - e^{-A\delta_0}w(t-\delta_0)|||_\beta \le \\
&\int_0^{\delta_0} c(\delta_0 - s)^{-\beta}\, \|A^{1/2}w(t-\delta_0+s)\|\ |||w(t-\delta_0+s)|||_\beta\, ds = \\
&|||w(t)|||_\beta \int_0^{\delta_0} c(\delta_0-s)^{-\beta}\, ||A^{1/2}w(t-\delta_0+s)|| \frac{|||w(t-\delta_0+s)|||_\beta}{|||w(t)|||_\beta}\, ds \le \\
&\frac{c\delta_0^{1-\beta}}{1-\beta} t^{\kappa/2-1/2}|||w(t)|||_\beta \le t^{\kappa/2-1/2}|||w(t)|||_\beta.
\end{aligned} \tag{60}$$

Similarly, using now the inequality $\|A^{3/4}w\| \le \|w\|^{1/4}\|Aw\|^{3/4}$, we have

$$\begin{aligned}
&|||w(t) - e^{-A\delta_0}w(t-\delta_0)|||_{1/2} \le \\
&\int_0^{\delta_0} c(\delta_0-s)^{-1/2}\|A^{1/2}w(t-\delta_0+s)\|\ |||w(t-\delta_0+s)|||_{3/4}\, ds \le \\
&|||w(t)|||_{1/2} \int_0^{\delta_0} c(\delta_0-s)^{-1/2}\frac{|||w(t-\delta_0+s)|||_{1/2}}{|||w(t)|||_{1/2}}\ t^{\kappa/4-3/4}\, ds \le \\
&c\delta_0^{1/2}t^{\kappa/4-3/4}|||w(t)|||_{1/2} \le t^{\kappa/4-3/4}|||w(t)|||_{1/2}
\end{aligned} \tag{61}$$

and also

$$|||w(t) - e^{-A\delta_0}w(t-\delta_0)|||_{1/2} \le t^{\kappa/4-3/4}|||w(t+\delta_0)|||_{1/2}/2. \tag{62}$$

It follows from (62) that

$$|||e^{-A\delta_0}w(t) - e^{-2A\delta_0}w(t-\delta_0)|||_{1/2} \le t^{\kappa/4-3/4}|||w(t+\delta_0)|||_{1/2}/2 \tag{63}$$

Beginning now with (58), we can also get

$$\begin{aligned}
&|||e^{-A\delta_0}w(t) - w(t+\delta_0)|||_{1/2} \le \\
&\int_0^{\delta_0} c(\delta_0-s)^{-1/2}\|A^{1/2}w(t+s)\|||||w(t+s)|||_{3/4}\, ds \le \\
&|||w(t+\delta_0)|||_{1/2} \int_0^{\delta_0} c(\delta_0-s)^{-1/2}\frac{|||w(t+s)|||_{1/2}}{|||w(t+\delta_0)|||_{1/2}}\ t^{\kappa/4-3/4}\, ds \le \\
&c\delta_0^{1/2}t^{\kappa/4-3/4}|||w(t+\delta_0)|||_{1/2} \le t^{\kappa/4-3/4}|||w(t+\delta_0)|||_{1/2}/2.
\end{aligned} \tag{64}$$

Thus, (63) and (64) yield that

$$|||e^{-2A\delta_0}w(t-\delta_0) - w(t+\delta_0)|||_{1/2} \le t^{\kappa/4-3/4}|||w(t+\delta_0)|||_{1/2}. \tag{65}$$

If we now apply gradually (61), Corollary 7, (60) and (65), we obtain

$$\begin{aligned}
&\frac{|||w(t-\delta_0)|||_\beta}{|||w(t)|||_{1/2}(1-t^{\kappa/4-3/4})} \ge \frac{|||w(t-\delta_0)|||_\beta}{|||e^{-A\delta_0}w(t-\delta_0)|||_{1/2}} \ge \\
&\frac{|||e^{-A\delta_0}w(t-\delta_0)|||_\beta}{|||e^{-2A\delta_0}w(t-\delta_0)|||_{1/2}} \ge \frac{|||w(t)|||_\beta(1-t^{\kappa/2-1/2})}{|||w(t+\delta_0)|||_{1/2}(1+t^{\kappa/4-3/4})}
\end{aligned}$$

and (57) immediately follows.

Let us fix now $t > 3$ and suppose that

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta_0)|||_{1/2}} = C \geq D_0. \tag{66}$$

We will show that this inequality leads to a contradiction. If we use (57) for this t we get

$$\frac{|||w(t-\delta_0)|||_\beta}{|||w(t)|||_{1/2}} \geq C\frac{(1-t^\theta)^2}{(1+t^\theta)}.$$

There exists $n \in N$ such that

$$2 < t - n\delta_0 \leq 3 < t - (n-1)\delta_0. \tag{67}$$

If we use (57) gradually for $t-\delta_0, t-2\delta_0, \dots t-(n-1)\delta_0$, we end up with

$$\frac{|||w(t-n\delta_0)|||_\beta}{|||w(t-(n-1)\delta_0)|||_{1/2}} \geq C\prod_{k=1}^{n}\frac{(1-(t-(k-1)\delta_0)^\theta)^2}{(1+(t-(k-1)\delta_0)^\theta)} =: C\eta.$$

Using now the elementary properties of the function $x \to \ln(1+x)$ and (67), we get

$$\ln\eta = \sum_{k=1}^{n} 2\ln(1-(t-(k-1)\delta_0)^\theta) - \ln(1+(t-(k-1)\delta_0)^\theta) >$$
$$\sum_{k=1}^{n} -4(t-(k-1)\delta_0)^\theta - (t-(k-1)\delta_0)^\theta = -\frac{5}{\delta_0}\sum_{k=1}^{n}\delta_0(t-(k-1)\delta_0)^\theta >$$
$$-\frac{5}{\delta_0}\int_1^\infty t^\theta\,dt = \frac{5}{(\theta+1)\delta_0}.$$

Therefore, $\eta > e^{\frac{5}{(\theta+1)\delta_0}}$ and

$$\frac{|||w(t-n\delta_0)|||_\beta}{|||w(t-(n-1)\delta_0)|||_{1/2}} > Ce^{\frac{5}{(\theta+1)\delta_0}}. \tag{68}$$

It follows now from (54), (67) and (68) that $C_0' > Ce^{\frac{5}{(\theta+1)\delta_0}}$, which together with (66) and (56) gives the following contradiction:

$$D_0 \leq C < C_0' e^{-\frac{5}{(\theta+1)\delta_0}} = D_0.$$

Therefore, we proved that if $t \geq 3$ then

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta_0)|||_{1/2}} \leq D_0.$$

Now it is possible to use the integral representation (58) and show that there exists a constant c_1 dependant only on the solution w and δ_0 such that

$$\frac{|||w(t+\delta)|||_{1/2}}{|||w(t)|||_{1/2}} < c_1,\ \forall t > 2 \text{ and } \forall\delta \in [0,\delta_0].$$

Therefore, we can write

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_{1/2}} = \frac{|||w(t)|||_\beta}{|||w(t+\delta_0)|||_{1/2}}\frac{|||w(t+\delta_0)|||_{1/2}}{|||w(t+\delta)|||_{1/2}} \leq D_0c_1,\ \forall t \geq 3 \text{ and } \forall\delta \in [0,\delta_0]. \tag{69}$$

If we use now the inequalities $|||w|||_{1/2}^2 \leq |||w|||_\beta|||w|||_{1-\beta}$ and $|||w|||_{1-\beta}^2 \leq 2|||w|||_{2-2\beta}||w||$ (which follow from the moment inequality), we finally obtain (using also (69) and (18)) that

$$\frac{|||w(t)|||_\beta}{\|w(t+\delta)\|} = \frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_{1/2}}\frac{|||w(t+\delta)|||_{1/2}}{|||w(t+\delta)|||_{1-\beta}}\frac{|||w(t+\delta)|||_{1-\beta}}{\|w(t+\delta)\|} \leq$$
$$2D_0c_1\frac{|||w(t+\delta)|||_\beta}{|||w(t+\delta)|||_{1/2}}\frac{|||w(t+\delta)|||_{2-2\beta}}{|||w(t+\delta)|||_{1-\beta}} \leq 2D_0^2c_1^2\frac{|||w(t+\delta)|||_{2-2\beta}}{|||w(t+\delta)|||_{1-\beta}} \leq$$
$$cD_0^2c_1^2\frac{|||w(t+\delta)|||_{1/2}}{|||w(t+\delta)|||_{1-\beta}} \leq cD_0^2c_1^2\frac{|||w(t+\delta)|||_\beta}{|||w(t+\delta)|||_{1/2}} \leq cD_0^3c_1^3, \forall t \geq 3 \text{ and } \forall\delta \in [0,\delta_0].$$

We set $C_1 = cD_0^3 c_1^3$ and the proof of Theorem 4 is complete. ○

Proof of Corollary 5: Let the assumptions of Corollary 5 be satisfied. As was discussed in the second section, w is a global strong solution on $[T_0, \infty)$. By the application of Theorems 3 and 4 we get immediately the assertion of Corollary 5. ○

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