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A FINITE DIFFERENCE SCHEME TO APPROXIMATE THE SOLUTION OF A TWO POINT BVP FOR A SEMILINEAR EQUATION

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Abstract: *In this article we propose a numerical method to approximate the solution of the following two point boundary value problem*

$$-u''(t) + \lambda \cdot u(t) + f(u(t)) = g(t), t \in (0,1), \quad u(0) = u(1) = 0$$

where f satisfies a Lipschitz condition and λ is a real number. The proposed scheme approximates the second order derivative by central finite difference formula and the discrete solution is computed using an iterative method.

Keywords: *semilinear periodic problem, maximal monotone operator, strongly positive operator, Lipschitz operator, Banach fixed point theorem*
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1 Introduction

We consider the semilinear periodic problem

$$-u''(t) + \lambda \cdot u(t) + f(u(t)) = g(t), t \in (0,1) \tag{1}$$

$$u(0) = u(1) = 0, \tag{2}$$

where $f : \mathbb{R} \to \mathbb{R}$ satisfies the Lipshitz condition $|f(x) - f(y)| \leq \alpha |x - y|$ for all $x, y \in \mathbf{R}$ ($\alpha > 0$) and $f(0) = 0$. We assume that the free term $g \in L^2(0,1)$ and the positive parameter λ satisfies the condition $\lambda > \alpha$.

The problem $(1), (2)$ is motivated by the logistic equation of population dynamics, or by the vibration of a string with self-interaction, and has been investigated by many authors(we refer for example to [1], [3], [4], [6], [7]).

In [1], the problem $(1), (2)$ is studied using the Green function, the nonlinear term f satisfying the following conditions:

(A1) f is a function of C^1 and $f(0) = f'(0) = 0$,
(A2) $h(u) := f(u)/u$ is strictly increasing ($h(0) = 0$).

In [3] the nonlinearity f is decreasing, satisfies an inequality of the type $|f(u)| \leq a\,|u|^p + b$, the free term g is continuous and $\lambda = 0$.

In the paper [4] is considered the problem $u''(t) + u(t) + f(u(t)) = g(t)$; $t \in (0, \pi)$; $u(0) = u(\pi) = 0$, where $g \in C([0, \pi])$, $f \in C(\mathbf{R})$ and moreover, the nonlinearity f satisfies:

(B1) $-\infty < f(-\infty) < f(x) < f(\infty) < \infty$ for all $x \in \mathbf{R}$, and

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(B2)
$$f(-\infty)\int_0^\pi \sin t dt < \int_0^\pi g(t)\sin t dt < f(\infty)\int_0^\pi \sin t dt$$

($f(-\infty)$and $f(\infty)$ signify the limits of the function f at $-\infty$, respectively at ∞). It proves, using a result of topological degree, that this problem has a solution.

A particular case of the problem (1), (2) when $\lambda = 0$ and $f(u) = a^2 \sin u$ is studied in the monograph [5]. Using the Green function, the problem is equivalently transformed into an integral equation of Hammerstein type. The existence of the solution is proved using the theory of fixed points for compact mappings.

Many boundary-value problems for nonlinear ordinary differential equations of second order are studied in the monograph [2].

Thus, in the chapter 4, it is considered the following problem:

$$u''(t) + f(u(t)) = g(t)\ t \in (0,1) \quad u(0) = u(1) = 0$$

.

It proves that if $g \in C([0,1])$, $f : \mathbb{R} \to \mathbb{R}$ is continuous and

$$m^2\pi^2 < a \le \frac{f(t)-f(s)}{t-s} \le b < (m+1)^2\pi^2$$

for some $m \in \mathbf{N}$ and all $t, s \in \mathbf{R}$ with $t \neq s$, then the problem has a unique solution. In the proof, after an ingenious equivalent transformation of the problem, the Banach fixed point theorem is used.

A general problem of the form $u''(t) + f(t, u(t)) = 0$, $t \in (0,1)$; $u(0) = u(1) = 0$ is studied in the chapter 6 of the monograph [2], when $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(t,0) = 0$ and $f(t,\eta) - f(t,\xi) \ge -\alpha(\eta - \xi)$ for some $\alpha > 0$ and all $\eta > \xi \ge 0$. Furthermore, it exists the limit $\ell(t)$ of the function $f_1(t,\xi) = \frac{f(t,\xi)}{\xi}$ when $\xi \to \infty$, uniformly on $[0,1]$, $\ell(t) > -\alpha$ in $[0,1]$ and $\frac{\partial f(t,\xi)}{\partial \xi}$ exists and is continuous in $[0,1] \times [0,\infty)$. The problem is equivalently transformed into an integral equation, and then, the existence of the solution is proved using the fixed point properties of the increasing maps from a cone into itself.

In [10] is presented a theorem stating the existence and uniqueness of our problem. The next section will present a numerical approach to approximate this unique solution.

2 Numerical method

To approximate numerically the unique solution of the problem (1), (2) we will use the central finite difference for the second order derivative of a function. For a sufficiently smooth function u we have

$$\frac{u(t+h) - 2u(t) + u(t-h)}{h^2} = u''(t) + \varepsilon(h)$$

where $\varepsilon(h) = \frac{h^2}{12}u^{(4)}(\xi) \quad \xi \in (t-h, t+h)$. In the next paragraphs we will consider a set of equispaced points in $(0,1)$ denoted by

$$\Delta = (0 = t_0 < t_1 < ... < t_{n-1} < t_n = 1)$$

that is $t_i = \frac{1}{n}$. We will also use u_i for the approximation of the solution u in the point t_i. With the above notations we have

$$-\frac{u(t_i+h) - 2u(t_i) + u(t_i-h)}{h^2} + \varepsilon_i(h) + \lambda u(t_i) + f(u(t_i)) = g(t_i), \quad i = 1, 2, ..., n-1$$

For h sufficiently small we neglect the terms $\varepsilon_i(h)$ and we find out that the approximations will satisfy the following equations:

$$-\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}+\lambda u_i+f(u_i)=g_i, \quad i=1,2,...,n-1$$

with $g_i=g(t_i)$ and $u_0=u_n=0$. The above equations may be written in vectorial form

$$\mathbf{A}(\mathbf{u})+\lambda\mathbf{u}+\mathbf{F}(\mathbf{u})=\mathbf{G} \tag{2.1}$$

where $\mathbf{u}=(u_1,u_2,...,u_{n-1})^t$, $\mathbf{G}=(g(t_1),g(t_2),...,g(t_{n-1}))^t$, and the operators $\mathbf{A},\mathbf{F}:\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$ are defined by $\mathbf{F}(\mathbf{u})=(f(u_1),f(u_2),...,f(u_{n-1}))^t$, $\mathbf{A}(\mathbf{u})=A\cdot\mathbf{u}$ with

$$A=\frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0\\ -1 & 2 & -1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & -1 & 2 & -1\\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

In $\mathbb{R}^{n-1}$ we will consider the euclidean inner product $\langle u,v\rangle=\sum u_iv_i$ and the corresponding norm $||u||=\sqrt{\langle u,u\rangle}$. With respect to this inner product it is easy to prove that

$$\langle A\cdot\mathbf{u},\mathbf{u}\rangle=\mathbf{u}^t\cdot A\cdot\mathbf{u}=(u_1^2+(u_1-u_2)^2+\ldots+(u_{n-2}-u_{n-1})^2+u_{n-1}^2)/h^2\geq 0 \tag{2.2}$$

We consider now the operator $\mathbf{L}:\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$ defined as $L=\mathbf{A}+\lambda I_{n-1}$ and we obtain

$$\langle\mathbf{L}\cdot\mathbf{u},\mathbf{u}\rangle=\langle A\cdot\mathbf{u},\mathbf{u}\rangle+\lambda||\mathbf{u}||\geq\lambda||\mathbf{u}||_2$$

As in continuous case it is easy to prove that operator $\mathbb{F}$ is a Lipschitz operator in respect with the euclidian norm. We have

$$\begin{aligned}||\mathbf{F}(\mathbf{u})-\mathbf{F}(\mathbf{v})||^2=&\sum(\mathbf{F}(\mathbf{u})_i-\mathbf{F}(\mathbf{v})_i)^2=\sum(f(u_i)-f(v_i))^2\\ \leq&\alpha\sum(u_i-v_i)^2=\alpha^2||\mathbf{u}-\mathbf{v}||^2\end{aligned}$$

With these properties established, we are ready to give the theorem that states the existence and uniqueness of the solution for the nonlinear system of equation 2.1 with the unknown vector $\mathbf{u}$.

Theorem 2.1. *Let $f:\mathbb{R}\to\mathbb{R}$, $g:[0,1]\to\mathbb{R}$, and λ, α positive parameters such that*

1. $|f(x)-f(y)|\leq\alpha|x-y| \qquad \forall x,y\in\mathbb{R}$
2. $\lambda>\alpha$

Then the nonlinear system of equations 2.1 has a unique solution.

Proof. From 2.2 we see that operator $\mathbf{L}$ is a strongly positive linear operator. Moreover, from 2.2 and Cauchy inequality we have

$$||\mathbf{Lu}||\geq\lambda||\mathbf{u}|| \qquad \forall\mathbf{u}\in\mathbb{R}^{n-1}.$$

Consequently $\mathbf{L}$ is an invertible operator and

$$||\mathbf{L}^{-1}||\leq\frac{1}{\lambda}$$

Then, writing the nonlinear system 2.1 under the form

$$\mathbf{Lu}+\mathbf{Fu}=\mathbf{G}$$

and applying the operator $\mathbf{L}^{-1}$ we get

$$\mathbf{u} = \mathbf{L}^{-1}\mathbf{G} - \mathbf{L}^{-1}\mathbf{F}\mathbf{u} \tag{2.3}$$

Now, we denote by $\mathbf{T} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ the following operator

$$\mathbf{T} = -\mathbf{L}^{-1}\mathbf{F}\mathbf{u} + \mathbf{L}^{-1}\mathbf{G}$$

and our nonlinear system of equation takes the form $\mathbf{u} = \mathbf{T}\mathbf{u}$ and the entirely problem reduces to the study of the fixed points of the operator T. We have,

$$\begin{aligned}||\mathbf{T}\mathbf{u} - \mathbf{T}\mathbf{v}|| =& ||\mathbf{L}^{-1}\mathbf{F}\mathbf{u} - \mathbf{L}^{-1}\mathbf{F}\mathbf{v}|| = ||\mathbf{L}^{-1}(\mathbf{F}\mathbf{u} - \mathbf{F}\mathbf{u})|| \\ \leq& ||\mathbf{L}^{-1}|| \cdot ||\mathbf{F}\mathbf{u} - \mathbf{F}\mathbf{u}|| \leq \frac{\alpha}{\lambda}||\mathbf{u} - \mathbf{v}|| \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^{n-1}\end{aligned}$$

Since $\alpha < \lambda$ we deduce that $\mathbf{T}$ is a contraction on $\mathbb{R}^{n-1}$ and using the Banach fixed point theorem the operator $\mathbf{T}$ has an unique fixed point, and thus the proof is complete. □

We are also able to prove the continuous dependence of the solution of nonlinear system 2.1 with respect to the right hand side.

Theorem 2.2. *Let $i \in \{1, 2\}$ and $\mathbf{u}^{(i)}$ the unique solution of the nonlinear system*

$$\mathbf{A}(\mathbf{u}) + \lambda\mathbf{u} + \mathbf{F}(\mathbf{u}) = \mathbf{G}^{(i)}$$

where $\mathbf{G}^{(i)} \in \mathbb{R}^{n-1}$. Then we have

$$||\mathbf{u}^{(1)} - \mathbf{u}^{(2)}|| \leq \frac{1}{\lambda - \alpha}||\mathbf{G}^{(1)} - \mathbf{G}^{(2)}||$$

Proof. Using 2.3 we have successively

$$\begin{aligned}||\mathbf{u}^{(1)} - \mathbf{u}^{(2)}|| =& ||\mathbf{L}^{-1}\mathbf{G}^{(1)} - \mathbf{L}^{-1}\mathbf{F}\mathbf{u}^{(1)} - \mathbf{L}^{-1}\mathbf{G}^{(2)} + \mathbf{L}^{-1}\mathbf{F}\mathbf{u}^{(2)}|| \\ \leq& ||\mathbf{L}^{-1}(\mathbf{G}^{(1)} - \mathbf{G}^{(2)})|| + ||\mathbf{L}^{-1}(\mathbf{F}\mathbf{u}^{(1)} - \mathbf{F}\mathbf{u}^{(2)})|| \\ \leq& ||\mathbf{L}^{-1}|| \cdot ||\mathbf{G}^{(1)} - \mathbf{G}^{(2)}|| + ||\mathbf{L}^{-1}|| \cdot ||\mathbf{F}\mathbf{u}^{(1)} - \mathbf{F}\mathbf{u}^{(2)}|| \\ \leq& \frac{1}{\lambda}||\mathbf{G}^{(1)} - \mathbf{G}^{(2)}|| + \frac{\alpha}{\lambda}||\mathbf{u}^{(1)} - \mathbf{u}^{(2)}||\end{aligned}$$

It results that

$$(\lambda - \alpha)||\mathbf{u}^{(1)} - \mathbf{u}^{(2)}|| \leq ||\mathbf{G}^{(1)} - \mathbf{G}^{(2)}||$$

that represents the conclusion of the theorem. □

Now, if the right hand side is smooth enough to assure the continuity of the forth order derivative of the exact solution we may prove the convergence of the approximate solution to the exact solution when h tends to zero. Indeed, if we denote by $[\mathbf{u}]_h$ the vector obtained by projecting the exact solution of the initial BVP on the discrete set of points Δ and $\mathbf{u}_h$ the approximate solution we have

$$\mathbf{L}\mathbf{u}_h + \mathbf{F}\mathbf{u}_h = \mathbf{G}$$

$$\mathbf{L}[\mathbf{u}]_h + \mathbf{F}[\mathbf{u}]_h = \mathbf{G} - \varepsilon(h)$$

where $\varepsilon(h) = (\varepsilon_1(h), \varepsilon_2(h), \ldots, \varepsilon_{n-1}(h))^t$. From the previous theorem we obtain

$$||[\mathbf{u}]_h - \mathbf{u}_h|| \leq \frac{1}{\lambda - \alpha}||\varepsilon(h)||$$

Since

$$||\varepsilon(h)|| = \sqrt{\sum_{i=1}^{n-1} \varepsilon_i(h)^2} = \sqrt{\sum_{i=1}^{n-1} \left(\frac{h^2}{12} u^{(iv)}(\xi_i)\right)^2}$$
$$\leq \frac{h^2}{12} M \sqrt{n-1} = \frac{h^2}{12} M \sqrt{\frac{1}{h} - 1}$$

where $M = \sup_{x\in[0,1]} |u^{(iv)}(x)|$, it follows that $||[\mathbf{u}]_h - \mathbf{u}_h|| = O(h^{3/2})$ and this proves the convergence of the numerical scheme when h tends to 0.

3 Numerical results

Our implementation follows the usual succesive approximation algorithm to find the unique fixed point of a cotraction. We start with the null vector as initial iteration ($\mathbf{u}^{(0)} = (0, 0, ..., 0)^t$) and compute the succesive iterations by formula

$$\mathbf{u}^{(s+1)} = \mathbf{T}(\mathbf{u}^s) = \mathbf{L}^{-1}(\mathbf{G} - \mathbf{F}(\mathbf{u}^s)), \qquad s \geq 0$$

We stop the iteration when the difference between two succesive iterations becomes less than a prescribed tolerance. As we have seen in the previous section, operator $\mathbf{L}$ is linear and has the associated matrix equal to $A + \lambda I_{n-1}$ which is Teoplitz (elements on the same diagonal are equals) tridiagonal and symmetric. Instead of computing the inverse of this matrix we will solve at every step s the linear system

$$\mathbf{L}\mathbf{u}^{(s+1)} = \mathbf{G} - \mathbf{F}(\mathbf{u}^s), \qquad s \geq 0 \tag{3.1}$$

In this way, the storage space is reduced considerably. The linear system 3.1 is solved using a LU decomposition of the matrix $A+\lambda I_{n-1}$. This decomposition is done before starting the iterations. To store the elements of the lower triunghiular matrix we need two vectors while for the upper triunghiular matrix we need one vector. All computation are done in $O(n)$ operation. Then the solution of the system 3.1 is obtained by succesive substitutions which also require only $O(n)$ arithmetic operations.

Example In this example we take the nonlinear term $f(x) = \ln\sqrt{1+x^2}$, $\lambda = 1$ and we choose the right hand side such that the exact solution of the problem to be $u(x) = x^3(1-x^3)$ that is $g(x) = -6\,x + 30\,x^4 + x^3 - x^6 + 1/2\,\ln\left(1 + (x^3 - x^6)^2\right)$. Since $f'(x) = \frac{x}{x^2+1} \leq \frac{1}{2}$ the Lipschitz condition is a straight consequence of the Lagrange's theorem. Figure 1 shows in the left part the exact and approximate solution on a grid with 15 equispaced points. In the right is presented the errors of approximation. The iterative process (fixed point algorithm) stoped after 10 iterations with the last two iteration being equal. Table 1 presents the evolution of the difference of two successive iterations.

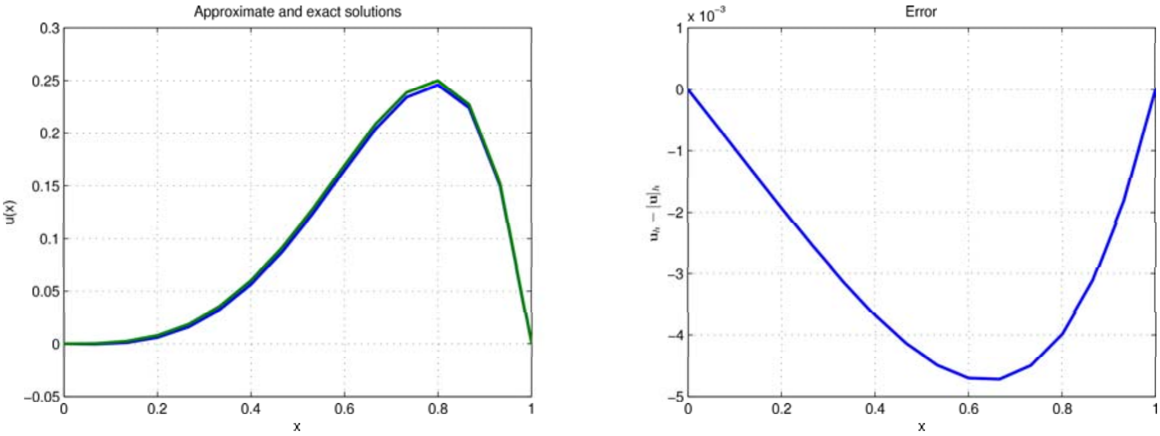
s	$r_s = \|\|\mathbf{u}^{(s+1)} - \mathbf{u}^{(s)}\|\|$	s	$r_s = \|\|\mathbf{u}^{(s+1)} - \mathbf{u}^{(s)}\|\|$	s	$r_s = \|\|\mathbf{u}^{(s+1)} - \mathbf{u}^{(s)}\|\|$
1	0.5362	4	$4.6647 \cdot 10^{-7}$	7	$7.3240 \cdot 10^{-13}$
2	0.0033	5	$5.4269 \cdot 10^{-9}$	8	$7.6732 \cdot 10^{-15}$
3	$3.9891 \cdot 10^{-5}$	6	$6.3067 \cdot 10^{-11}$	9	$1.8934 \cdot 10^{-16}$

Table 1:

Approximating the exact solution with the last iteration we get an error having the norm equal to 0.0129. Table 2 shows how the error of approximation depends on the step size of the grid. For each value of h we considered as approximate solution the tenth iteration in the successive approximation algorithm.

h	0.1	0.05	0.01	0.005	0.001
$e_h = \|\|[\mathbf{u}]_h - \mathbf{u}_h^{(10)}\|\|$	0.0236	0.0084	$7.5046 \cdot 10^{-4}$	$6.7126 \cdot 10^{-5}$	$2.3733 \cdot 10^{-5}$

Table 2:



(a) Approximate and exact solutions

(b) Error

Figure 1: Comparison between approximate and exact solutions

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