REVISITING THE POPOVICIU-BOHMAN-KOROVKIN THEOREM

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Abstract: *We present the history of the theorem, including the contribution of Tiberiu Popoviciu and Harald Bohman of the 50's; we present its applications in the domain of the uniform approximation in $C([a,b])$ and some modern points of vue.*

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1 A few of history

The starting point of all the story of the uniform approximation was the moment of 1885 when the famous mathematician Karl Weierstrass (1815-1897), already old, gave his celebrated approximation theorem:

Theorem 1. (Karl Weierstrass, 1885) *Any continuous function $f:[a,b]\to\mathbb{R}$ is the uniform limit of a sequence of polynomial functions $(P_n)_n$ (with real coefficients).*

This deep result was such exciting that later other mathematicians gave new proofs of this theorem. We mention the proof of Charles de la Vallée-Poussin, Henri Lebesgue (1898) (see [28]), Tiberiu Popoviciu in 1950 (see [23]), Dimitrie D. Stancu in 1959 (see [25]).

A very important moment of the finding of the proofs for the theorem of Weierstrass was the one produced by Serge Bernstein in 1912, when he constructed, associated to every function $f\in C([0,1])$, and for any $n\in\mathbb{N}$, the polynomial

$$(B_n f)(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right).$$

The sequence of these polynomials is uniformly convergent to f. Note that S. Bernstein used a proabilistic method to obtain his theorem (this method was deeply developed by Dimitrie D. Stancu in a fundamental work [27]). (As known, the polynomials of interpolation of Lagrange, $L_n f$, does not converge always to f if $n\to\infty$, cf. C. Méray, C. Runge, S. N. Bernstein, G. Faber, G. Grünwald, J. Marcinkiewicz.)

Of course, the polynomial of Bernstein conducted to an operator

$$B_n: C([a,b])\to C([a,b]),$$

called the operator of Bernstein and very studied.

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Another approximation operator was constructed by L. Féjer, starting from the interpolation polynomial of Hermite H_{2n+1} with all the nodes only double and choosing these nodes being the roots of the polynomial of Tchebycheff T_{n+1}. So, this approximation gave a new uniform convergence to f and then, a new proof of the theorem of Weierstrass.

But after these remarkable examples, to construct a new approximation operator was not easy; although, note the works of T. Popoviciu 1932 ([22]), G. M. Mirakyan 1941 ([19]), J. Favard 1944 ([8]), O. Szász 1950 and V. A. Baskakov 1957 ([3]).

2 The results of T. Popoviciu, H. Bohman and P. P. Korovkin

Come now in the mathematics of the '50 s.

The romanian mathematician Tiberiu Popoviciu, a former member of the Romanian Academy, presents in a Conference given at this Institution in 1950, two short works ([23] and [24]) in which he deeply analyses the conditions for the uniform convergence of a sequence of positive linear operators $(L_n)_n$

$$L_n : C([a,b]) \to C([a,b]),$$

to the identity operator,

$$L_n f \xrightarrow[(n\to\infty)]{(unif)} .f$$

In another work of 1952, [5], Harald Bohman finds that, if $(A_n)_n$ is a sequence of linear and positive operators

$$A_n : C([0,1]) \to C([0,1])$$

and

$$\begin{aligned} &A_n(1) \to 1 \\ &A_n(x) \to x \\ &A_n(x^2) \to x^2 \end{aligned}$$

uniformly on $[0,1]$, then $A_n f \to f$, uniformly on $[0,1]$, for any $f \in C([0,1])$.

This result and the one of T. Popoviciu were examined by P. P. Korovkin in 1953, which formulated it in a very clear manner (see [9]). Six years later, in 1959, he published the book entitled „Linear Operators and Approximation Theory" in Russian, that was translated in English and published in 1960 by Hindustan Publishing Corp. of Delhi (India) (see [10]).

This theorem is, in a modern version, the following:

Theorem 2. *Let $L_n : C([a,b]) \to C([a,b])$ be a sequence of positive linear operators and let $e_k(x) = x^k$, $k = 0,1,2,\ldots$. If $L_n(e_k)$ converges uniformly (for $n \to \infty$) to e_k on $[a,b]$ for $k = 0,1$ and 2, then the sequence $L_n(f)$ converges uniformly to f on $[a,b]$, for each $f \in C([a,b])$.*

And so the theorem was mentioned as „The Theorem of Bohman-Korovkin" or, more often, as „The Theorem of Korovkin". But the most correct name can be considered „The Theorem of Popoviciu-Bohman-Korovkin"

We can consider this fact as a mistake in the appreciation of the results of the romanian mathematicians.

It exists a trigonometric version (using 1, $\cos x$, $\sin x$) of the theorem; we don't discuss this here.

3 Some examples of using the theorem

For any positive linear operator (more precisely, sequence of operators) $L_n : C([a,b]) \to C([a,b])$, by the theorem, it is sufficient to compute the action of the operator L_n on the

„testing-functions“ e_0, e_1 and e_2 called the moments of L_n; if these converge to e_0, e_1, respectively e_2, uniformly, then the uniform convergence $(L_n f) \xrightarrow[(n\to\infty)]{} f$ is obtained.

We give some examples.

(a) The operator of S. Bernstein

We obtain

$$\begin{cases} B_n e_0 = e_0 \\ B_n e_1 = e_1 \\ (B_n e_2)(x) = x^2 + \dfrac{x(1-x)}{n} \xrightarrow[(n\to\infty)]{(\text{unif.})} x^2, \end{cases}$$

which gives again the well-known result.

(b) The operator of D. D. Stancu (see [26])

This operator is given by the formula

$$\left(S_n^{\langle\alpha\rangle} f\right)(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[m,-\alpha]}} f\left(\frac{k}{m}\right),$$

where

$$\begin{cases} y^{[n,h]} = y(y-h)(y-2h)\cdot\ldots\cdot(y-(n-1)h) \quad \text{if} \quad n \geq 1 \\ y^{[0,h]} = 1 \end{cases}$$

(of course, if $\alpha = 0$, we find again the operator of Bernstein).

We have

$$\begin{cases} S_n^{\langle\alpha\rangle} e_0 = e_0 \\ S_n^{\langle\alpha\rangle} e_1 = e_1 \\ \left(S_n^{\langle\alpha\rangle} e_2\right)(x) = x^2 + \dfrac{x(1-x)(1+m\alpha)}{m(1+\alpha)} \longrightarrow e_2(x). \end{cases}$$

(c) The operator of binomial type T. Popoviciu

A sequence of polynomials $(p_n)_n$ is said to be of binomial type if every p_n is of degree n and the equalities

$$p_n(u+v) = \sum_{k=0}^{n} \binom{n}{k} p_k(u) p_{n-k}(v)$$

are satisfied identically in u and v, for any non-negative integer n (that gives immediately $p_0 = 1$ and $p_n(0) = 0$ for any $n \geq 1$).

Suppose that $p_n(1) \neq 0$, for any $n \in \mathbb{N}$. Then one can define the linear operator of Tiberiu Popoviciu (see [22])

$$(T_n f)(x) = \frac{1}{p_n(1)} \sum \binom{n}{k} p_k(x) p_{n-k}(x) f\left(\frac{k}{m}\right)$$

(for any $f \in C([0,1])$).

This operator was studied by C. Manole in his thesis [18] (under the guidance of D. D. Stancu), by means of umbral calculus (finite operatorial calculus). We have

$$\begin{cases} T_n e_0 = e_0 \\ T_n e_1 = e_1 \\ (T_n e_2)(x) = x^2 + \dfrac{x(1-x)}{n} + x(1-x) a_n^{(2)}, \end{cases}$$

where $a_n^{(2)}$ are certain coefficients which depend to n and tend to 0, where $n \to \infty$. So $T_n e_2 \xrightarrow[(n\to\infty)]{} e_2$ (uniformly).

We mention that there is also a bivariate form of the criterion, that can be used to study the convergence of the positive linear bivariate operators of approximation (see [30]).

For many details on the theorem of Korovkin (Popoviciu-Bohman-Korovkin) see [1], [2], [7], [12], [17], [29], [31], [32].

4 Some contemporary remarks

A modern point of vue on the theorem Korovkin is presented in [11]; it is, in fact, a generalization of the theorem in the space

$$C(X) = \{f : X \to \mathbb{R} \mid f \quad \text{continuous}\},$$

where X is a compact space, endowed with the norm

$$||f||_\infty = \sup_{x \in X} |f(x)|.$$

The classical result are found again.

A deep relationship between the theorem of Korovkin and the absolute continuity is given in [21]. In fact, in [21] is given an explanation of the phenomenon of the theorem of Korovkin by the absolute continuity.

A version of the theorem of Korovkin (Popoviciu-Bohman-Korovkin) for weighted approximation are given in [6]. Here is also given an example of space in which, for a certain weighted approximation there is no a Korovkin finite subset (of the type $\{e_0, e_1, e_2, \ldots, e_k\}$). This is linked to the fact that the Čech compactification of $[0, \infty)$ is not metrizabile (recently proved by Woods (see [33])).

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References

[1] Agratini, O., *Aproximarea prin operatori liniari*, Presa Universitară clujeană, 2000.
[2] Altomare, F., Campiti, M., *Korovkin-Type Approximation Theory and its Applications*, Walter de Gruyter, Berlin, 1994.
[3] Baskakov, V. A., *An example of sequence of linear positive operators in the space of continuous functions*, Docl. Akad. Nauk. S.S.S.R, **113** (1957), 249-254.
[4] Bernstein, S. N., *Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités*, Commun. Soc. Math. Kharkow (2), **13** (1912-13), 1-2.
[5] Bohman, H., *On approximation of continuous and of analytic functions*, Ark. Math., **2** (1952), 43-56.
[6] Bustamente, J., L. Morales de la Cruz, *Korovkin Tpe Theorems for Weighted Approximation*, Int. J. of Math. Analysis, vol. 1, 2007, no. 26, 1273-1283.
[7] Cheney, E. W., *Introduction in Approximation Theory*, Chelsea Publishing Company, New York, N. Y., 1982.
[8] Favard, F., *Sur les multiplicateurs d'interpolation*, J. Math. Press Appl., **23** (9) (1944), 219-247.
[9] Korovkin, P. P., *On the convergence of linear positive operators in the space of continuous functions*, (Russian), Docl. Akad. Nauk. S.S.S.R, (N.S), **90** (1953), 961-964.
[10] Korovkin, P. P., *Linear Operators and Approximation Theory*, Hindustan Publishing Coorp., Delhi (India), 1960 (translated from Russian Edition of 1959).

[11] Loméli, H. E., Garsia, C. L., *Variations on a Theorem of Korovkin*, Amer. Math. Monthly, vol. **13** (2006), 744-750.
[12] Lorentz, G. G., *Approximation of Functions*, Chelsea Publishing Company, New York, N. Y., 1986.
[13] Lupaş, A., *The approximation by means of some linear positive operators*, IDoMAT 95, Dortmund, Math. Research, vol. **86** (1995), 201-229.
[14] Lupaş, A., *Classical polynomials and approximation theory*, Kolloquium Vortrag, Univ. Duisburg, Dec. 1986.
[15] Lupaş, A., *Approximation operators of binomial type*, Proc. of IDoMAT 98, Birkhäuser Verlag, Witten (1998).
[16] Lupaş, L., Lupaş, A., *Polynomials of binomial type and approximation*, Studia Univ. Babeş-Bolyai, Mathematica, **32** (1987), 61-69.
[17] Lupaş, L., *Teoria constructivă a funcţiilor*, Editura Universităţii din Sibiu, 1994.
[18] Manole, C., *Dezvoltări în serii de polinoame Appell generalizate cu aplicaţii la aproximarea funcţiilor*, teză de doctorat, Cluj, 1987.
[19] Mirakyan, G. M., *Aproximation of continuous functions with the aid of polynomials*(Russian), Docl. Akad. Nauk. S.S.S.R, **31** (1941), 201-205.
[20] Natanson, I. P., *Constructive Function Theory*, Frederik Ungar Publishing Co. New York, 1964.
[21] Niculescu, C. P., *An overview of absolute continuity and its applications*, în vol. „Inequalities and Applications“ (Proc. of the Conf. Ineq. and Appl. '07) Int. Series of Numerical Math., vol. 157, 201-214, Birkhäuser Verlag, 2008.
[22] Popoviciu, T., *Remarques sur les polynômes binomiaux*, Bul. Soc. Sci., Cluj, **6** (1931), 146-148.
[23] Popoviciu, T., *Asupra formei restului în unele formule de aproximaţie ale analizei*, în „Lucrările sesiunii generale ştiinţifice a Acad. R.P.R.“, iunie 1950, 193-186.
[24] Popoviciu, T., *Asupra demonstraţiei teoremei lui Weierstrass cu ajutorul polinoamelor de interpolare*, în „Lucrările sesiunii ştiinţifice a Acad. R.P.R.“, 1950, 1664-1667.
[25] Stancu, D. D., *Asupra unei demonstraţii a teoremei lui Weierstrass*, în „Bul. Inst. Politehnic din Iaşi“, t. 5, fasc. 1-2, 1959, 47-50.
[26] Stancu, D. D., *Approximation of functions by a new class of linear polynomial operators*, Rev. Roum. Math. Pures et Appl., **13** (1968), No. 2, 31-45.
[27] Stancu, D. D., *Use of probabilistic methods in the theory of uniform approximation of continuous functions*, Rev. Roum. Math. Pures et Appl., **14** (1969), No. 5, 673-691.
[28] Stancu, D. D., *Curs şi culegere de Probleme de Analiză numerică*, Universitatea „Babeş-Bolyai“, Cluj-Napoca, 1977.
[29] Stancu, D. D., Coman, Gh., Agratini, O., Trâmbiţaş, R., *Analiză numerică şi teoria aproximării*, vol. 1, Presa Universitară Clujeană, Cluj-Napoca, 2001.
[30] Stancu, D. D., Vernescu, A., *Approximation of bivariate functions by means of a class of operators of Tiberiu Popoviciu type*, Mathematical Reports (formerly Studii şi Cercetări Matematice), vol. 1 (51), (1999), No. 3, 411-420.
[31] Uchiyama, M., *Korovkin-type theorem for Schwarz maps and operator monotone functions in* $\mathbb{C}^*$ *algebras*, Math. Z. **230** (1999), 785-797.
[32] Uchiyama, M., *Proof of Korovkin's theorem via inequality*, Amer. Math. Montly, **110** (2003), 334-336.
[33] Woods, R. G., *The minimum uniform compactification of a metric space*, Fundam. Math., **147** (1995), 39-53.

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