JOURNAL OF SCIENCE AND ARTS

Z-VALUED FUNCTION ON SEMIGROUPS, Z-VALUED SESQUILINEAR FORMS

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Abstract: *We shall formulate and study some boundedness conditions for a Z–valued kernel on a semigroup and for Z–valued positive definite functions on $*$–semigroups. Then sesquilinear Z–valued forms and $\mathcal{F}(\mathcal{H}, Z)$–valued kernels are introduced, by analogy to the sesquilinear forms, replacing $\mathbb{C}$ by the admissible space in the Loynes sense Z. It is shown that the set $\mathcal{F}(\mathcal{H}, Z)$ it is a $*$–linear space which it is endowed with a positive cone $\mathcal{F}_+(\mathcal{H}, Z)$ induced by the positive cone from Z. A theorem of Kolmogorov–Aronszajn type is studied for positive definite $\mathcal{F}(\mathcal{H}, Z)$–valued kernels.*

Keywords: *Loynes spaces, kernels, $*$-semigroups*
2000 Mathematics subject classification: *47A45; 42B10*

1. Introduction

We recall see [1], the definition of an admissible space. A locally convex space Z is called *admissible in the Loynes sense* if the following five conditions are satisfied: Z is complete; there is a closed convex cone in Z, denoted Z_+, that defines an order relation on Z (that is $z_1 \leq z_2$ if $z_2 - z_1 \in Z_+$); there is an involution in Z, $Z \ni z \to z^* \in Z$ (that is $z^{**} = z$, $(\alpha z)^* = \overline{\alpha} z^*$, $(z_1 + z_2)^* = z_1^* + z_2^*$), such that $z \in Z_+$ implies $z^* = z$; the topology of Z is compatible with the order (that is there exists a basis of convex solid neighbourhoods of the origin); and any monotonously decreasing sequence in Z_+ is convergent.

Let Z be an admissible space in the Loynes sense. A linear topological space $\mathcal{H}$ is called *pre-Loynes Z–space* if satisfies the following properties:

$\mathcal{H}$ is endowed with an Z–valued *inner product* (gramian), i.e. there exists an application $\mathcal{H} \times \mathcal{H} \ni (h, k) \to [h, k] \in Z$ having the properties:

$[h, h] \geq 0$; $[h, h] = 0$ implies $h = 0$; $[h_1 + h_2, h] = [h_1, h] + [h_2, h]$; $[\lambda h, k] = \lambda[h, k]$; $[h, k]^* = [k, h]$ for all $h, k, h_1, h_2 \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. The topology of $\mathcal{H}$ is the weakest locally convex topology on $\mathcal{H}$ for which the application $\mathcal{H} \ni h \to [h, h] \in Z$ is continuous. Moreover, if $\mathcal{H}$ is a complete space with this topology, then $\mathcal{H}$ is called *Loynes Z– space*.

It is known, see [1], that if p is a continuous and monotonous seminorm on Z, then $q_p(h) = (p([h, h]))^{1/2}$ is a continuous seminorm on $\mathcal{H}$.

Also, by [1], if $\mathcal{H}$ is a pre-Loynes Z-space and $\mathcal{P}$ is a set of monotonous (increasing) seminorms defining the topology of Z, then the topology of $\mathcal{H}$ is defined by the sufficient and directed set of seminorms $Q_{\mathcal{P}} = \{q_p \mid p \in \mathcal{P}\}$.

Furthermore in [1], for every monotonous seminorm p on Z we have:

Paper presented at The VII-th International Conference on Nolinear Analysis and Applied Mathematics (ICNAAM), Târgovişte, 26-27 june, 2009

$p([h,k]) \le 2q_p(h)\cdot q_p(k)$ for all $h,k \in \mathcal{H}$.

We suppose that $mq_{p_2}(x) \le q_{p_1}(x) \le Mq_{p_2}(x)$, $(\forall)x \in \mathcal{H}$, with p_1, p_2 continuous and increasing seminorms on Z and M finite, $M \ge m > 0$. Then,

$$q_{p_2}^2\left(\frac{x}{q_{p_2}(x)} + \frac{y}{q_{p_2}(y)}\right) \le \frac{1}{m^2} q_{p_1}^2\left(\frac{x}{q_{p_2}(x)} + \frac{y}{q_{p_2}(y)}\right) =$$

$$= \frac{1}{m^2} p_1\left(\left[\frac{x}{q_{p_2}(x)} + \frac{y}{q_{p_2}(y)}, \frac{x}{q_{p_2}(x)} + \frac{y}{q_{p_2}(y)}\right]\right) = \frac{1}{m^2} p_1\left(\frac{[x,x]}{q_{p_2}^2(x)} + \frac{[y,y]}{q_{p_2}^2(y)} + \frac{[x,y]+[y,x]}{q_{p_2}(x)q_{p_2}(y)}\right) \le$$

$$\le \frac{1}{m^2}\left\{\frac{q_{p_1}^2(x)}{q_{p_2}^2(x)} + \frac{q_{p_1}^2(y)}{q_{p_2}^2(y)} + p_1\left(\frac{[x,y]+[y,x]}{q_{p_2}(x)q_{p_2}(y)}\right)\right\} \le \frac{1}{m^2}\left\{2M^2 + \frac{p_1([x,y]+[y,x])}{q_{p_2}(x)q_{p_2}(y)}\right\} \le$$

$$\le \frac{1}{m^2}\left\{2M^2 + M^2\frac{p_1([x,y]+[y,x])}{q_{p_1}(x)q_{p_1}(y)}\right\} = \frac{M^2}{m^2}\left\{2 + \frac{p_1([x,y]+[y,x])}{q_{p_1}(x)q_{p_1}(y)}\right\}$$

$$q_{p_2}^2\left(\frac{x}{q_{p_2}(x)} + \frac{y}{q_{p_2}(y)}\right) = p_2\left(\left[\frac{x}{q_{p_2}(x)} + \frac{y}{q_{p_2}(y)}, \frac{x}{q_{p_2}(x)} + \frac{y}{q_{p_2}(y)}\right]\right)$$

$$\left[\frac{x}{q_{p_2}(x)} + \frac{y}{q_{p_2}(y)}, \frac{x}{q_{p_2}(x)} + \frac{y}{q_{p_2}(y)}\right] =$$

$$= \left[\frac{x}{q_{p_2}(x)}, \frac{x}{q_{p_2}(x)}\right] + \left[\frac{y}{q_{p_2}(y)}, \frac{y}{q_{p_2}(y)}\right] + \left[\frac{x}{q_{p_2}(x)}, \frac{y}{q_{p_2}(y)}\right] + \left[\frac{y}{q_{p_2}(y)}, \frac{x}{q_{p_2}(x)}\right] \ge$$

$$\ge 2\left\{\left[\frac{x}{q_{p_2}(x)}, \frac{y}{q_{p_2}(y)}\right] + \left[\frac{y}{q_{p_2}(y)}, \frac{x}{q_{p_2}(x)}\right]\right\} = 2\frac{[x,y]+[y,x]}{q_{p_2}(x)q_{p_2}(y)}.$$

Because p_2 is increasing, we have

$$q_{p_2}^2\left(\frac{x}{q_{p_2}(x)} + \frac{y}{q_{p_2}(y)}\right) \ge 2\frac{p_2([x,y]+[y,x])}{q_{p_2}(x)q_{p_2}(y)}$$

Thus,

$$2\frac{p_2([x,y]+[y,x])}{q_{p_2}(x)q_{p_2}(y)} \le \frac{M^2}{m^2}\left\{2 + \frac{p_1([x,y]+[y,x])}{q_{p_1}(x)q_{p_1}(y)}\right\},$$

or

$$\frac{p_2([x,y]+[y,x])}{q_{p_2}(x)q_{p_2}(y)} \le \frac{M^2}{m^2}\left\{1 + \frac{1}{2}\frac{p_1([x,y]+[y,x])}{q_{p_1}(x)q_{p_1}(y)}\right\}.$$

We recall by the Definition 1.1, [2], that a (pre-) Loynes Z–space $\mathcal{H}$, consisting of Z–valued functions on Λ, admits reproducing kernel, if there exists a positive definite kernel $\Gamma = \Gamma_{\mathcal{H}}$, which satisfies the following conditions:

(a) $\Gamma(\lambda,\cdot) \in \mathcal{H}$, for all $\lambda \in \Lambda$;

(b) $h(\lambda) = [h(\cdot), \Gamma(\lambda,\cdot)]$, for all $\lambda \in \Lambda$ and $h \in \mathcal{H}$.

(c) the closed subspace generated by $\Gamma(\lambda,\cdot)$, $\lambda \in \Lambda$ is accessible in $\mathcal{H}$.

2. Operatorial kernels

In this section $\mathcal{H}$ will be first a complex vector space, Z an admissible space in the Loynes sense and $\mathcal{F}(\mathcal{H}, Z)$ the set of Z–valued sesquilinear functions on $\mathcal{H}$, i.e. the set of operators B,

$$B : \mathcal{H} \times \mathcal{H} \to Z,$$

which satisfy

$$(1) \qquad \begin{cases} B(\alpha_1 h_1 + \alpha_2 h_2, k) = \alpha_1 B(h_1, k) + \alpha_2 B(h_2, k), \\ B(h, \beta_1 k_1 + \beta_2 k_2) = \overline{\beta}_1 B(h, k_1) + \overline{\beta}_2 B(h, k_2), \end{cases}$$

for any $h, h_j, k, k_j \in \mathcal{H}$; $\alpha_j, \beta_j \in \mathbb{C}$ $(j = 1, 2)$.

Putting $Z = \mathbb{C}$, we see that the elements of $\mathcal{F}(\mathcal{H}, \mathbb{C})$ are known as sesquilinear forms (or functionals). In analogy with this fact, the elements $B \in \mathcal{F}(\mathcal{H}, Z)$ will be called Z–*sesquiliniar forms*. It is obvious that $\mathcal{F}(\mathcal{H}, Z)$ is endowed in a natural way with a structure of linear space, every element of $\mathcal{F}(\mathcal{H}, Z)$ satisfying the parallelogram rule and a calculus rule by diagonally values

$$(2) \qquad B(h + k, h + k) + B(h - k, h - k) = 2[B(h, h) + B(k, k)]$$

$$B(h, k) = \sum_{j=0}^{3} i^j B(h + i^j k, h + i^j k)$$

with i–imaginary unit $h, k \in \mathcal{H}$.

Often we will suppose that $\mathcal{H}$ is endowed with a gramian such that $\mathcal{H}$ is a Loynes Z–space. In this case we will say that B is *continuous* if for every seminorm $p \in \mathcal{P}_Z$, there exists a constant $M_p > 0$ and two seminorms $p_1, p_2 \in \mathcal{P}_Z$ so that

$$(3) \qquad p(B(h, k)) \leq M_p q_{p_1}(h) q_{p_2}(k); \quad h, k \in \mathcal{H}.$$

Following, we will name the set of this Z–sesquilinear forms by $\mathcal{FC}(\mathcal{H}, Z)$.

Particularly we will say that Z–form $B \in \mathcal{F}(\mathcal{H}, Z)$ is q–bounded and we shall denote this by $B \in \mathcal{F}Q(\mathcal{H}, Z)$, if for any $p \in \mathcal{P}_Z$, there exists a constant $M_p > 0$ such that

$$(4) \qquad p\left(B(h, k)\right) \leq M_p q_p(h) q_p(k), \quad h, k \in \mathcal{H}.$$

If above we can choose M_p independent of $p \in \mathcal{P}_Z$, then we shall say that B is universally bounded and we shall denote $B \in \mathcal{FU}(\mathcal{H}, Z)$.

It is obvious that the above subclasses are linear subspaces in $\mathcal{F}(\mathcal{H}, Z)$ and satisfy the inclusions

$$(5) \qquad \mathcal{FU}(\mathcal{H}, Z) \subset \mathcal{F}Q(\mathcal{H}, Z) \subset \mathcal{FC}(\mathcal{H}, Z).$$

It was noticed that (4) take places if B satisfies a " q–boundedness condition" concerning to the gramian, i.e. for $p \in \mathcal{P}_Z$ there exists $N_p > 0$ such that

$$(6) \qquad p(B(h, k)) \leqq N_p p([h, k]), \quad h, k \in \mathcal{H}.$$

Indeed, applying in the right side the inequality $p([h, k]) \leq 2q_p(h) \cdot q_p(k)$ we shall obtain (4) with $M_p = 2N_p$. A similar fact take places if we impose a "universally boundedness " condition concerning to the gramian.

We shall say that Z– sesquilinear form B is *positive*, if satisfies

$$(7) \qquad B(h, h) \geqq 0, \quad h \in \mathcal{H}$$

and *symmetrical*, if

$$(8) \qquad B(h,k) = [B(k,h)]^*, \quad h,k \in \mathcal{H}.$$

Moreover, in $\mathcal{F}(\mathcal{H}, Z)$ we can introduce an involution using the involution of Z hereby:

$$(9) \qquad B^*(h,k) := [B(k,h)]^*, \quad h,k \in \mathcal{H}.$$

Indeed, it is easy to see that B^* thus defined is also a $Z-$ sesquilinear form, and the application

$$\mathcal{F}(\mathcal{H}, Z) \ni B \to B^* \in \mathcal{F}(\mathcal{H}, Z)$$

is an involution, i.e. satisfies

$$B^{**} = B$$

$$(\alpha_1 B_1 + \alpha_2 B_2)^* = \overline{\alpha}_1 B_1^* + \overline{\alpha}_2 B_2^*,$$

for any $B, B_1, B_2 \in \mathcal{F}(\mathcal{H}, Z)$ and $\alpha_1, \alpha_2 \in \mathbb{C}$.

In this context, Z–positive forms are self-adjoint elements and the symmetrical forms are exactly the self-adjoint elements from $\mathcal{F}(\mathcal{H}, Z)$. We use now for the first and the second class the notations: $\mathcal{F}_+(\mathcal{H}, Z)$, respectively $\mathcal{F}_h(\mathcal{H}, Z)$.

An another particular subclass in $\mathcal{F}(\mathcal{H}, Z)$ is the subclass $\mathcal{FB}(\mathcal{H}, Z)$ consisting of those Z–form B for which there exists a constant $M_B > 0$, such that $B(h,h) \leqq \mu_B[h,h]$, $h \in \mathcal{H}$. Because the last relation implies $B(h,h) = B(h,h)^*$ and $-\mu_B[h,h] \leq B(h,h)$ it results that these Z–forms are symmetrically and for its it has sense in analogy with the case of bounded operators, to consider the borders

$$m_B := \sup\{\mu > 0 : -\mu[h,h] \leq B(h,h),\ h \in \mathcal{H}\}$$

and

$$M_B := \inf\{\nu > 0 : B(h,h) \leq \nu[h,h],\ h \in \mathcal{H}\},$$

which are optimally with the property

$$m_B[h,h] \leq B(h,h) \leq M_B[h,h], \quad h \in \mathcal{H}.$$

A modality to obtain Z–sesquilinear forms on a Loynes Z–space $\mathcal{H}$ is given, as in the case of Hilbert spaces, using operators and inner product.

Indeed if $T \in \mathcal{L}(\mathcal{H})$, then defining

$$(10) \qquad B_T(h,k) := [h, Tk] \quad (h,k \in \mathcal{H})$$

we shall obtain that $B_T \in \mathcal{F}(\mathcal{H}, Z)$.

It is easy to observe that the application

$$(11) \qquad \mathcal{L}(\mathcal{H}) \ni T \to B_T \in \mathcal{F}(\mathcal{H}, Z)$$

is linear and one-to-one.

If $T \in \mathcal{L}^*(\mathcal{H})$ then the easy calculus

$$B_T^*(h,k) = [B_T(k,h)]^* = [k, Th]^* = [Th, k] = [h, T^*k] = B_{T^*}(h,k), \quad (h,k \in \mathcal{H}),$$

shows that the restriction to $\mathcal{L}^*(\mathcal{H})$ of $T \to B_T$ is an involution.

It is also easy to notice that $T \in \mathcal{L}^*(\mathcal{H})$, if and only if $B_T^* = B_S$ for a certain $S \in \mathcal{L}(\mathcal{H})$.

Indeed, the relation $B_T^* = B_S$ is equivalent with $[Th, k] = [h, Sk]$, for $h, k \in \mathcal{H}$ i.e. it is equivalent with $S = T^*$.

More, this observation allows us to establish that the application (11) isn't generally onto.

Indeed, if $T \in \mathcal{L}(\mathcal{H}) \setminus \mathcal{L}^*(\mathcal{H})$, then doesn't exist any $S \in \mathcal{L}(\mathcal{H})$ such that $B_S = B_T^*$, because otherwise we should have $S = T^*$.

Of course we can refer also on the restrictions of the application (11). For example if $T \in \mathcal{C}(\mathcal{H})$, by

$$q_p(Th) \leq M_p q_{p_0}(h), \quad (h \in \mathcal{H})$$

we have successively

$$p(B_T(h,k)) = p([h,Tk]) \leqq 2q_p(h)q_p(Tk) \leqq 2M_p q_p(h) q_{p_0}(k), \quad h,k \in \mathcal{H},$$

which means $B_T \in \mathcal{FC}(\mathcal{H}, Z)$. By the same reasoning on $T \in \mathcal{C}Q(\mathcal{H})$, having above $p_0 = p$, we will infer that B_T satisfies (4) where M_p is replaced with $2M_p$, i.e. $B_T \in \mathcal{F}Q(\mathcal{H}, Z)$.

By analogy: $T \in \mathcal{CU}(\mathcal{H}) \Rightarrow B_T \in \mathcal{FU}(\mathcal{H}, Z)$ and

$$T \in \mathcal{L}_+(\mathcal{H}) \Rightarrow B_T \in \mathcal{F}_+(\mathcal{H}, Z), \; T \in \mathcal{L}_h(\mathcal{H}) \Rightarrow B_T \in \mathcal{F}_h(\mathcal{H}, Z).$$

If we analyze also the boundedness relations for the forms from $\mathcal{F}B(\mathcal{H}, Z)$, we easily observe using the inequality from Consequence 1.1.1 (see [2]) or [1], that $T \in \mathcal{B}_h(\mathcal{H})$ implies $B_T \in \mathcal{FB}(\mathcal{H}, Z)$ and more, its borders coincide: $m_T = m_{B_T}$, $M_T = M_{B_T}$.

By analogy with the expression of norm of elements $T \in \mathcal{B}_h(\mathcal{H})$

$$\|T\| = \max\{|m_T|, |M_T|\}, \quad T \in \mathcal{B}_h(\mathcal{H})$$

is clear that $\mathcal{FB}(\mathcal{H}, Z)$ becomes norm space with

$$\|B\| := \max\{|m_B|, |M_B|\}, \quad B \in \mathcal{FB}(\mathcal{H}, Z).$$

Rejoining the above results, we can enunciate:

Theorem 2.1 (i) *Given a (complex) linear space $\mathcal{H}$ and a locally convex space Z admissible in the Loynes sense, the set $\mathcal{F}(\mathcal{H}, Z)$ of $Z-$ sesquilinear forms on $\mathcal{H}$ is a $*$-linear space, the involution is defined by* (9) *and is endowed with a positive cone $\mathcal{F}_+(\mathcal{H}, Z)$ induced of the positive cone in Z by* (7).

Besides, every Z-sesquilinear forms satisfies the rules (2).

(ii) *If $\mathcal{H}$ is a Loynes Z-space, then there exists a natural embedding of $\mathcal{L}(\mathcal{H})$ in $\mathcal{F}(\mathcal{H}, Z)$ given of* (10) *and* (11) *with the properties:*

(a) *for $T \in \mathcal{L}(\mathcal{H})$ we have that $T \in \mathcal{L}^*(\mathcal{H})$ if and only if $B_T^* = B_S$ for a $S \in \mathcal{L}(\mathcal{H})$, case when $S = T^*$;*

(b) *generally it isn't onto;*

(c) *its restriction to $\mathcal{L}^*(\mathcal{H})$ keeping the involution, consequently the positive and symmetrical elements;*

(d) *Its restrictions range to the subspaces $\mathcal{C}(\mathcal{H})$, $\mathcal{C}Q(\mathcal{H})$, $\mathcal{CU}(\mathcal{H})$, $\mathcal{B}_h(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ are contents in $\mathcal{FC}(\mathcal{H}, Z)$, respectively $\mathcal{F}Q(\mathcal{H}, Z)$, $\mathcal{FU}(\mathcal{H}, Z)$, $\mathcal{FB}(\mathcal{H}, Z)$ above definite.*

In particular the elements from $\mathcal{FB}(\mathcal{H}, Z)$ are symmetrical.

3. $\mathcal{F}(\mathcal{H}, Z)$–valued kernels

In this section we shall consider the kernels $\mathcal{F}(\mathcal{H}, Z)$–valued on an arbitrary set $\wedge$ and for this we shall prove a typical factorization theorem. But, first we shall give an example of such kernel.

Let $\mathcal{H}$ be a linear space, Z–an admissible space, $\wedge$ an arbitrary set and $\mathcal{F}(\mathcal{H}, Z)$ the space of Z–sesquilinear forms on $\mathcal{H}$ which is Z-valued.

A $\mathcal{F}(\mathcal{H}, Z)$–valued kernel on $\wedge$, $C : \wedge \times \wedge \to \mathcal{F}(\mathcal{H}, Z)$ is called positively defined if,

$$\sum_{j,l=1}^{n} C(s_j, s_l)(h_l, h_j) \geq 0 \text{ for any } n \in \mathbb{N},\ s_1, \ldots, s_n \in \wedge \tag{12}$$

and $h_1, \ldots, h_n \in \mathcal{H}$. It is obvious that this fact is equivalent with the fact that the associate Z–valued kernel Γ_C on $\wedge_1 = \wedge \times \mathcal{H}$ defined by $\Gamma_C(\lambda, \mu) = C(t, s)(h, k)$, where $\lambda = (s, h)$, $\mu = (t, k)$ is positively defined.

Example 3.1 Let $\mathcal{H}$ be a linear space, $\mathcal{K}$ a Loynes Z-space and a family of linear operators defined by

$$D : \wedge \to \mathcal{L}(\mathcal{H}, \mathcal{K}).$$

Then, it is easy to see that the kernel $C : \wedge \times \wedge \to \mathcal{F}(\mathcal{H}, Z)$ defined by $C(s, t)(h, k) := [D(t)h, D(s)k]_{\mathcal{K}}$ is $\mathcal{F}(\mathcal{H}, Z)$–valued and positively defined.

Indeed ,

$$\sum_{j,l=1}^{n} C(s_j, s_l)(h_l, h_j) = \sum_{j,l=1}^{n} [D(s_l)h_l, D(s_j)h_j]_{\mathcal{K}} = \left[\sum_{l=1}^{n} D(s_l)h_l, \sum_{j=1}^{n} D(s_j)h_j\right]_{\mathcal{K}} \geqq 0,$$

therefore takes place (12).

Moreover, if $\mathcal{H}$ is a Loynes Z–space and the function $D(t) \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, then $C(s, t) \in \mathcal{FC}(\mathcal{H}, Z)$ because

$$p\left(C(s, t)(h, k)\right) = p\left([D(t)h, D(s)k]_{\mathcal{K}}\right) \leq 2q_p(D(t)h)q_p(D(s)k) \leq 2M_p^1 M_p^2 \cdot q_{p_1}(h)q_{p_2}(k),$$

for any $p \in \mathcal{P}_Z$, $h, k \in \mathcal{H}$.

Similarly we can obtain that if D has values in $\mathcal{C}Q(\mathcal{H}, \mathcal{K})$, $\mathcal{CU}(\mathcal{H}, \mathcal{K})$ respectively $\mathcal{B}(\mathcal{H}, \mathcal{K})$, then associate kernel has values in $\mathcal{F}Q(\mathcal{H}, Z)$, $\mathcal{FU}(\mathcal{H}, Z)$, respectively $\mathcal{FB}(\mathcal{H}, Z)$.

Remark 3.1 If C is a positive definite kernel, then the following relations take place:

$$[C(s, t)]^* = C(t, s), \quad (s, t \in \wedge); \tag{13}$$

$$C(s, s) \geq 0 \text{ for any } s \in \wedge; \tag{14}$$

equality (13) being supposed in the sense of involution from $\mathcal{F}(\mathcal{H}, Z)$ defined in (9) and (14) in the sense of positivity from inequality (12).

More, the following inequality takes place

$$p^2[C(t, s)(h, k)] \leq 4p[C(s, s)(h, h)]p[C(t, t)(k, k)], \tag{15}$$

for any $p \in \mathcal{P}_Z$ and $h, k \in \mathcal{H}$.

Proof. If C is a positive definite kernel then the Z-valued kernel Γ_C pe $\wedge_1 = \wedge \times \mathcal{H}$ is positively defined and applying the Proposition 3.1.1 (see [2]) we obtain $\Gamma_C(\lambda, \lambda) = C(s,s)(h,h) \geqq 0$ for any $\lambda = (s,h) \in \wedge_1$ and $\Gamma_C(\lambda,\mu)^* = \Gamma_C(\mu,\lambda)$, $(\lambda, \mu \in \wedge)$ i.e. $[C(t,s)(h,h)]^* = C(s,t)(k,h)$ or $C(s,t)^*(k,h) = C(s,t)(k,h)$ for any $h, k \in \mathcal{H}$. More, the inequality

$$p^2(\Gamma_C(\lambda,\mu)) \leqq 4p(\Gamma_c(\lambda,\lambda))p(\Gamma_C(\mu,\mu)),\ (\lambda,\mu \in \wedge,\ p \in \mathcal{P}_Z)$$

becomes

$$p^2(C(t,s)(h,k)) \leq 4p(C(s,s)(h,h))p(C(t,t)(k,k))$$

(where $\lambda = (s,h)$, $\mu = (t,k)$). □

We also use a part of Theorem 3.1.3, see [2]. Given a Z–valued positive definite kernel $\Gamma : \Lambda \times \Lambda \to Z$, there exists a Loynes space $\mathcal{H}$ and a function $f : \Lambda \to \mathcal{H}$ such that $\Gamma(\lambda,\mu) = [f(\lambda), f(\mu)]_{\mathcal{H}}$; $\lambda, \mu \in \Lambda$. In addition, $\mathcal{H}$ can be chosen to satisfy the minimality property, $\bigvee_{\lambda \in \Lambda} f(\lambda) = \mathcal{H}$.

Now the following theorem of Kolmogorov–Aronszajn type takes place. See [4] for Hilbert case.

Theorem 3.1 *Let C be a $\mathcal{F}(\mathcal{H}, Z)$–valued positive definite kernel on $\wedge$ ($\mathcal{H}$ being a linear space). Then, there exists a Loynes Z–space $\mathcal{K}$ and a function $D : \wedge \to \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that*

$$C(s,t)(h,k) = [D(t)h, D(s)k]_{\mathcal{K}} \quad (h,k \in \mathcal{H},\ s,t \in \wedge), \tag{16}$$

$$\mathcal{K} = \vee\{D(t)\mathcal{H} : t \in \wedge\}. \tag{17}$$

Proof. We define $\Gamma = \Gamma_C : (\wedge \times \mathcal{H}) \times (\wedge \times \mathcal{H}) \to Z$ by

$$\Gamma_C(\lambda,\mu) = C(t,s)(h,k),\ \text{unde}\ \lambda = (s,h),\ \mu = (t,k). \tag{19}$$

Because

$$\sum_{j,l=1}^{n} \Gamma(\lambda_j, \lambda_l) = \sum_{j,l=1}^{n} C(s_l, s_j)(h_j, h_l) \geqq 0$$

C is a $\mathcal{F}(\mathcal{H}, Z)$–valued kernel (positive definite), it results that Γ is a Z–valued positive definite kernel. Now, applying the Theorem 3.1.3, (see [2]), for Γ, there exists a Loynes Z–space $\mathcal{K}$ and a function $f : \wedge \times \mathcal{H} \to \mathcal{K}$ so that

$$\Gamma_C(\lambda,\mu) = [f(\lambda), f(\mu)]_{\mathcal{K}}, \quad \lambda,\mu \in \wedge \times \mathcal{H}, \tag{20}$$

$$\mathcal{K} = \bigvee_{\lambda \in \wedge \times \mathcal{H}} f(\lambda). \tag{21}$$

Since (19) and (20), considering $f(t,h)$ with (t,h) arbitrarily in $\wedge \times \mathcal{H}$, will result:

$$[f(s, \alpha_1 h_1 + \alpha_2 h_2), f(t,h)]_{\mathcal{K}} = \Gamma_C((s, \alpha_1 h_1 + \alpha_2 h_2), (t,h)) =$$
$$= C(t,s)(\alpha_1 h_1 + \alpha_2 h_2, h) = \alpha_1 C(t,s)(h_1, h) + \alpha_2 C(t,s)(h_2,h) =$$
$$= \alpha_1 \Gamma_C((s,h_1),(t,h)) + \alpha_2 \Gamma_C((s,h_2),(t,h)) = \alpha_1 [f(s,h_1), f(t,h)]_{\mathcal{K}} +$$
$$+ \alpha_2 [f(s,h_2), f(t,h)]_{\mathcal{K}} = [\alpha_1 f(s,h_1) + \alpha_2 f(s,h_2), f(t,h)]_{\mathcal{K}},$$

whence applying (21) we have $f(s,\cdot) \in \mathcal{L}(\mathcal{H},\mathcal{K})$ for any $s \in \wedge$.

We denote with D the function $\wedge \ni t \mapsto f(t,\cdot) \in \mathcal{L}(\mathcal{H},\mathcal{K})$. Thus $D(t) : \mathcal{H} \to \mathcal{K}$ given of $D(t)h = f(t,h)$ satisfies (16) from

$$C(s,t)(h,k) = [f(\lambda), f(\mu)]_{\mathcal{K}} = [f(t,h), f(s,k)]_{\mathcal{K}} = [D(t)h, D(s)k]_{\mathcal{K}},$$

where $\lambda = (t,h)$, $\mu = (s,k) \in \wedge \times \mathcal{H}$.. Since (21) results (17):

$$\mathcal{K} = \bigvee_{\lambda \in \wedge \times \mathcal{H}} f(\lambda) = \vee\{D(t)\mathcal{H} : t \in \wedge\}.$$

□

Now, we separately formulate the factorization theorem for particular cases when the considered kernel C is $\mathcal{G}(\mathcal{H},Z)$–valued with $\mathcal{G}$ in one of the assumptions:

$$\mathcal{FC},\ \mathcal{FB}.$$

Theorem 3.2 *If the positive definite kernel C from previous theorem is*

(i) $\mathcal{FC}(\mathcal{H},Z)$*–valued, denoting with $M_p(s)$ and $p_1(s)$, $p_2(s)$ the positive constant and the seminorms which appear in the condition* (3) *for Z–form $C(s,s)$ ($s \in \wedge$) associate to $p \in \mathcal{P}_Z$, then the operators $D(s)$ ($s \in \wedge$) from factorization theorem belong to $\mathcal{C}(\mathcal{H},\mathcal{K})$ and satisfies*
$$q_p^{\mathcal{K}}(D(s)h) \leqq [M_p(s)]^{1/2} q_{p_3}^{\mathcal{H}}(h), \quad h \in \mathcal{H},$$
where $p_3 \in \mathcal{P}_Z$, $p_3 \geqq \max\{p_1, p_2\}$,

(ii) $\mathcal{FB}(\mathcal{H},Z)$*–valued, then $D(s) \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and*
$$\|D(s)\| = \|C(s,s)\|^{1/2}, \quad s \in \wedge,$$
where $\|C(s,s)\|$ is the norm of Z–positive form $C(s,s)$, introduced by Theorem 2.1.

Proof. We recall that $q_p^{\mathcal{K}}(k) = \{p([k,k]_{\mathcal{K}})\}^{1/2}$ and $q_p^{\mathcal{H}}(h) = \{p([h,h]_{\mathcal{H}})\}^{1/2}$, the conclusions (i), (ii) shall be obtained by a careful examination of appropriate relations and its transcription in the seminorms language from $\mathcal{H}$, respectively $\mathcal{K}$ generated by elements from $\mathcal{P}_Z$ using gramian. □

In the $\mathcal{F}(\mathcal{H},Z)$–valued positive definite kernels class we distinguish two interesting subclasses. First particularizing Z–forms from $\mathcal{F}(\mathcal{H},Z)$ as in (11), we obtain the $\mathcal{L}(\mathcal{H})$–valued positive definite kernels on an arbitrary set $\wedge$ and then particularizing $\wedge$ to a semigroup S (or exactly $*$–semigroup with or without unit) we shall obtain $\mathcal{B}(\mathcal{H},Z)$–valued positive definite kernels on the semigroup S.

By the general previous results, for positive definite operator kernels ($\mathcal{L}(\mathcal{H})$–valued) on an arbitrary set $\wedge$, we deduce the following factorization theorem:

Theorem 3.3 (i) *If $\mathcal{H}$ is a Loynes Z–space and $T : \wedge \times \wedge \to \mathcal{L}(\mathcal{H})$ is a positive definite kernel on $\wedge$, then the following take place*

(a) *T is $\mathcal{L}^*(\mathcal{H})$–valued and $T(t,s)^* = T(s,t)$, $s,t \in \wedge$;*

(b) *$T(s,s) \in \mathcal{L}_+(\mathcal{H})$, $s \in \wedge$;*

(c) *there exists a Loynes* Z*–space* $\mathcal{K}$ *and a function*

$$D : \wedge \to \mathcal{L}^*(\mathcal{H}, \mathcal{K}) \text{ such that } T(s,t) = D^*(t)D(s), \;\; s,t \in \wedge$$

(ii) *If, in particular, the positive definite kernel* T *on* $\wedge$ *is*

(a) $\mathcal{C}(\mathcal{H})$*–valued, then its values are in* $\mathcal{C}(\mathcal{H}) \cap \mathcal{C}^*(\mathcal{H})$*, the previous relations have a corresponding transposition and the operators* $D(t)$ *from the minimal factorization are from* $\mathcal{C}^*(\mathcal{H}, \mathcal{K})$*;*

(b) $\mathcal{B}(\mathcal{H})$*–valued, then* $T(s,t) \in \mathcal{B}^*(\mathcal{H})$*,* $s,t \in \wedge$*,* $T(s,s) \in \mathcal{B}_+(\mathcal{H})$*,* $s \in \wedge$ *and* $D(t) \in \mathcal{B}^*(\mathcal{H}, \mathcal{K})$*,* $t \in \wedge$*.*

Proof. (i) Because, as such was specified for the $\mathcal{L}(\mathcal{H})$–valued kernels, the kernel T is positively defined if the kernel $\mathcal{F}(\mathcal{H}, Z)$–valued B_T is positively defined, according to Remark 3.1 by the relation (13) we have

$$B_{T(s,t)} = B^*_{T(t,s)}, \quad s,t \in \wedge$$

and this fact leads successively to

$$[h, T(s,t)k] = B_{T(s,t)}(h,k) = B^*_{T(t,s)}(h,k) = [B_{T(t,s)}(k,h)]^* =$$

$$= [k, T(t,s)h]^* = [T(t,s)h, k]$$

for any $h, k \in \mathcal{H}$. It results that there exists $T(t,s)^*$ and coincides with $T(s,t)$. Hereby we have (a). (b) is a consequence of (14) and of the properties of the application (11) from the Theorem 2.1.

For (c) we apply the Theorem (of factorization) 3.1 for the kernel $B_{T(\cdot,\cdot)}$ and we obtain the relation

$$[h, T(s,t)k] = B_{T(s,t)}(h,k) = [D(t)h, D(s)k], \quad s,t \in \wedge, \;\; h,k \in \mathcal{H}$$

which shows that $h \mapsto [D(t)h, D(s)k]$ admits Riesz representation also there exists $D(t)^*$ which satisfies the relation

$$T(s,t)k = D(t)^*D(s)k, \quad k \in \mathcal{H}, \;\; s,t \in \wedge.$$

(ii) will be checked applying the Theorem 3.2 □

Now we can give some boundedness conditions for a Z–valued kernel on a semigroup.

Definition 3.1 *Let* S *be an abelian semigroup and* $\Gamma : S \times S \to Z$ *a* Z*–valued kernel on* S*.* Γ *satisfies the boundedness condition, if there is a function* $c : S \to [0, \infty)$ *so that*

(BC) $$c(u)\Gamma - \Gamma_u$$

is positive definite for all $u \in S$*, where* $\Gamma_u(s,t) := \Gamma(us, ut)$*.*

Γ *will satisfy the "continuity" condition* (CC)*, if for every seminorm* $p \in \mathcal{P}_Z$*, there exist two functions on* S*,* $\gamma_p : S \to \mathcal{P}_Z$ *and* $c_p : S \to [0, \infty)$ *such that*

(CC) $$p\left(\sum_{j,k=1}^{n} c_j \bar{c}_k \Gamma_u(s_j, s_k)\right) \le c_p(u)\gamma_p(u)\left(\sum_{j,k=1}^{n} c_j \bar{c}_k \Gamma(s_j, s_k)\right)$$

for all $n \in \mathbb{N}$*,* $c_1, \ldots, c_n \in \mathbb{C}$*,* $s_1, \ldots, s_n, u \in S$*.*

Definition 3.2 *If C is a $\mathcal{F}(\mathcal{H},Z)$–valued kernel on the semigroup S, then C satisfies*
(i) *the boundedness condition, if there exists a function*
$\rho : S \to [0,\infty)$ *such that*

$$\text{(BC)} \qquad \rho(u)C - C_u \text{ is positively defined } (u \in S),$$

where

$$C_u(s,t) := C(us, ut);$$

(ii) *the "continuity" condition* (CC), *if there exist the functions $c_p : S \to [0,\infty)$ and $\gamma_p : S \to \mathcal{P}_Z$ such that takes place the condition* (CC) *from the Definition* 3.1 *with Γ_C and Γ_{C_u} instead of Γ, and Γ_u respectively.*

Now, we shall focus our attention on the functions $\mathcal{F}(\mathcal{H},Z)$–valued of positive type on $*$–semigroups.

Definition 3.3 *The function $\mathcal{F}(\mathcal{H},Z)$–valued ϕ defined on $*$–semigroup S is named positively defined if the $\mathcal{F}(\mathcal{H},Z)$–valued associate kernel $C_\phi : S \times S \to \mathcal{F}(\mathcal{H},Z)$ defined by $C_\phi(s,t) := \phi(t^*s)$, $s,t \in S$ is positively defined.*
We say that such of function ϕ satisfies the boundedness conditions (BC), (CC) *if the associate kernel C_ϕ satisfies the corresponding conditions from the Definition* 3.2.

Considering now the Remark 3.1, we can formulate

Consequence 3.1 *Every $\mathcal{F}(\mathcal{H},Z)$–valued function ϕ positive definite on $*$–semigroup S satisfies the relations (with involution and positivity from $\mathcal{F}(\mathcal{H},Z)$)*

$$\text{(26)} \qquad \phi(s^*s) \geqq 0,$$

$$\phi(s)^* = \phi(s^*), \quad s \in S,$$
$$\left[p\left(\phi(t^*s)\right)\right]^2 \leqq 4p(\phi(s^*s))p(\phi(t^*t)).$$

$$\text{(27)} \qquad \left[p\left(\sum_{j,l=1}^{n} \phi(t_l^* s_j)(h_j k_l)\right)\right]^2 \leqq 4p\left(\sum_{j,l=1}^{n} \phi(s_l^* s_j)(h_j, h_l)\right) p\left(\sum_{j,l=1}^{n} \phi(t_l^* t_j)(h_j, k_l)\right),$$

for any $p \in \mathcal{P}_Z$, $s,t \in S$, $\overline{s} = (s_1,\dots,s_n) \subset S$, $\overline{t} = (t_1,\dots,t_n) \subset S$, $\overline{h} := (h_1,\dots,h_n)$, $\overline{k} := (k_1,\dots,k_n) \subset \mathcal{H}$.

Remark 3.2 *$\mathcal{F}(\mathcal{H},Z)$–valued kernel C_ϕ associate with the $\mathcal{F}(\mathcal{H},\Gamma)$–valued function ϕ on $*$–semigroup S, satisfies the transfer property (CT):*

$$C_\phi(us,t) = \phi(t^*us) = \phi\left((u^*t)^*s\right) = C_\phi(s,u^*t); \; u,s,t \in S.$$

Now, taking into consideration the Theorem 3.1 of factorization, we deduce,

Corollary 3.1 *If $\phi : S \to \mathcal{F}(\mathcal{H},Z)$ is a positive definite function on the $*$–semigroup S, then there exists a Loynes Z–space $\mathcal{K}$ and a function $D : S \to \mathcal{L}(\mathcal{H},\mathcal{K})$ such that*

$$\text{(30)} \qquad \phi(t^*s)(h,k) = [D(t)h, D(s)k]_{\mathcal{K}}; \; h,k \in \mathcal{H}, \; s,t \in S$$

(31) $$\mathcal{K} = \vee\{D(t)\mathcal{H},\ t \in S\}.$$

Remark 3.3 *If the function ϕ from above is $\mathcal{G}(\mathcal{H}, Z)$–valued, with $\mathcal{G}$ in one of positions $\mathcal{FC}$, $\mathcal{FB}$, then the function $D(\cdot)$ takes values in $\mathcal{C}(\mathcal{H}, \mathcal{K})$ and $\mathcal{B}(\mathcal{H}, \mathcal{K})$ respectively.*

Using again the previous results (Theorem 3.2, Theorem 3.3), we obtain:

Remark 3.4 *If the function ϕ is operatorial valued ($\mathcal{L}(\mathcal{H})$–valued), then from the conditions of Consequence 3.1, the factorization have the following form*

$$\phi(t^*s) = D(t^*)D(s), \quad (s, t \in S) \text{ with } D(s) \in \mathcal{L}(\mathcal{H}, \mathcal{K}),$$

with determination that, if we have successively $\phi(s) \in \mathcal{C}(\mathcal{H})$, $\mathcal{B}(\mathcal{H})$, then in a corresponding way

$$D(s) \in \mathcal{C}(\mathcal{H}, \mathcal{K}),\ \mathcal{B}(\mathcal{H}, \mathcal{K}), \quad (s \in S).$$

References

[1] Ciurdariu, L., *On the topology of Loynes spaces*, Bull. Şt. al U.P.T. Seria Matem-Fizică, Tom 49(63), 2, (2004), pp. 52-59.

[2] Ciurdariu, L., *Classes of linear operators on pseudo-Hilbert spaces and applications*, Part I and II, Tipografia Universitatii de Vest Timisoara, 2006, 2008.

[3] Gaspar, P., *Analiza armonica pe spaţii de variabile aleatoare*, Universitatea de Vest, Timişoara, 2008.

[4] Stochel, J., *Dilability of sesquilinear form-valued kernels*, Ann. Polon. Math. XLVIII, 1988, 1-30.

Manuscript received: 16.06.2009 / accepted: 22.08.2009