SELF-PROPULSION OF THIN PROFILES. AN ANALYTIC APPROACH

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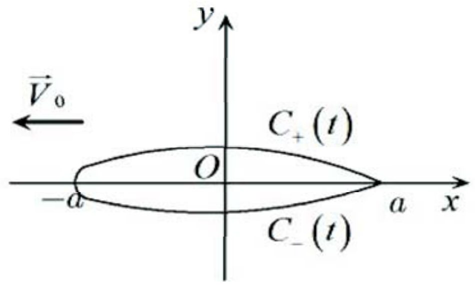
Abstract: *The paper deals with the incompressible flow past oscillating thin profiles. In the frame work of the linearized theory, the pressure jump over the oscillatory profile is the solution of a hypersingular integral equation. Using an asymptotic expansion of the kernel with respect to the frequency of the oscillationo (which is a small parameter) and keeping the leading terms, one obtaines a simplified form of the integral equation. We consider the case of the flate profile and after solving the integral equation we calculate the aerodynamic coefficients. For some particular profiles, the oscillatory motion can determine the apparition of a propulsive force,when average drag coefficient becomes negative.*

Keywords: *ear, model, system, optimal, quadratic*

1 Formulation of the Problem

An incompressible unsteady flow past an infinite cylindrical body moving with a given velocity is considered. Unperturbed state is characterized by the fields of velocity, $-V_0\mathbf{i}$, the pressure p_∞, and the constant density ρ_0. For an arbitrary time t, perturbation is given by the pressure field, $p(\mathbf{x}, t)$, and velocity $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$.



With respect to the of ε - thickness body, only thin profiles are considered, in a such way, perturbation produced over the fluid will be linearly. Term of the order $\mathcal{O}(\varepsilon)$ are leading. Boundary conditions will be imposed over the chord of the profile which in this case can be taken as projection of the profile onto the Ox-axis.

Results found in this paper are based on the general theory described in [5] and follows as technique approaching, the paper [1], [2].

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2 The Integral Equation for the Pressure Jump

Perturbation $(\mathbf{v}, p)$ produced by an unsteady incompressible flow past a thin profile is described by the following equations

$$\operatorname{div}\mathbf{v} = 0, \quad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot \operatorname{grad})\,\mathbf{v} + \frac{1}{\rho_0}\operatorname{grad} p = 0 \tag{1}$$

with respect to a Cartesian framework Oxy, where $\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = u\mathbf{i} + v\mathbf{j}$, $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$.

Let $C = C(t)$ be - the profile of equation $F(x, y, t) \equiv y - h(x, t) = 0$. Denote by $\mathbf{n} = (n_x, n_y, n_t)$, normal vector to the wing profile at arbitrary point, (x, y, t), then

$$\mathbf{n} = (n_x, n_y, n_t) = \left(-\frac{\partial h}{\partial x},\ 1,\ -\frac{\partial h}{\partial t}\right) \tag{2}$$

For our sake, we suppose that wing profile is thin enough such that perturbations produced into the fluid are very small and one may neglect products of perturbations quantities. By following, from Eqs. (1) ca be linearized equations around the rest state and the equations of motion in term of distributions are found as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial y} = f\,\delta_C \tag{3}$$

where $f = \frac{1}{\rho_0}[p]_C$ with $[p]_C$ is the prssure jump across the profile. Since

$$\Delta \mathcal{E}_2 = \delta(\mathbf{x}, t) \iff \mathcal{E}_2 = \frac{\delta(t)}{2\pi}\ln|\mathbf{x}|$$

from Eq. (3) we get the pressure field and the downwash, in terms of distributions

$$p(\mathbf{x}, t) = \rho_0 \frac{\partial}{\partial y}\left(\frac{\delta(t)}{2\pi}\ln|\mathbf{x}|\right) * (f\,\delta_C) \tag{4}$$

$$v(\mathbf{x}, t) = -H(t)\frac{\partial^2}{\partial x^2}\left(\frac{1}{2\pi}\ln|\mathbf{x}|\right) * f\,\delta_C \tag{5}$$

where $H(t)$ is Heaviside function. Therefore, for $y \neq 0$ one takes place the integral representation of the downwash distribution

$$v(x, y, t) = -\frac{1}{2\pi}\int_{-\infty}^{t} \mathrm{d}t' \int_{C(t')} f(x', t')\frac{\partial^2}{\partial y^2}\ln\sqrt{(x - x')^2 + y^2}\mathrm{d}x' \tag{6}$$

where $C(t')$ is projection of the lifting profile onto Ox' - axis at the arbitrary moment t'.

Let consider $Ox^{(1)}y^{(1)}$ a new system of coordinates related to the profile that has an uniform translation moving of velocity $-V_0\mathbf{i}$ with respect to the Oxy system coordinates, where

$$x^{(1)} = x + V_0 t, \quad y^{(1)} = y \tag{7}$$

Introduce also $s^{(1)}$ - the spatial variable by

$$s^{(1)} = x^{(1)} - x'^{(1)} - V_0(t - t'). \tag{8}$$

With respect to the new coordinates, the integral representation (6) becomes, for $y^{(1)} \neq 0$,

$$v\left(x^{(1)}, y^{(1)}, t\right) =$$

$$= -\frac{1}{2\pi V_0}\int_{-\infty}^{x^{(1)}-x'^{(1)}} \mathrm{d}s^{(1)} \int_{-a}^{a} f\left(x'^{(1)}, t\right) \frac{\partial^2}{\partial y^{(1)^2}} \ln\sqrt{s^{(1)^2}+y^{(1)^2}}\mathrm{d}x'^{(1)} \tag{9}$$

We consider that lifting profile is subjected to harmonic oscillations and set

$$\begin{aligned} &h\left(x^{(1)}, t\right) = h\left(x^{(1)}\right)\exp\left(i\omega t\right),\ f\left(x^{(1)}, t\right) = f\left(x^{(1)}\right)\exp\left(i\omega t\right), \\ &v\left(x^{(1)}, y^{(1)}, t\right) = d^{(1)}\left(x^{(1)}, y^{(1)}\right)\exp\left(i\omega t\right). \end{aligned} \tag{10}$$

then, pass to the dimensionless coordinates by

$$(x, y, s) = \left(\frac{x^{(1)}}{a}, \frac{y^{(1)}}{b}, \frac{s^{(1)}}{a}\right) \tag{11}$$

and, for simplicity, keep (x, y) as notation for the spatial coordinates of a fixed point belongs to the wing profile.

Also denote profile projection on the Oxy - plane with respect to the dimensionless coordinates. Then, one may replace (11) into Eq.(9) and take the limit when $y \longrightarrow 0$ whence, for $|x| \leq 1$ the two dimensional hypersingular integral equation for the pressure jump states

$$\frac{1}{2\pi V_0}\int_{-1}^{1} f\left(a\xi\right)\exp\left\{-i\frac{\omega a}{V_0}\left(x-\xi\right)\right\}\mathrm{d}\xi\int_{-\infty}^{x-\xi}\exp\left(i\frac{\omega}{V_0}as\right)\frac{\mathrm{d}s}{s^2} = -d\left(x\right). \tag{12}$$

In Eq.(12), $\rho_0\,\mathrm{Re}(f\exp(i\omega t))$ is the jump of the pressure over the oscillating wing, ρ_0 is the density of the fluid at rest, ω is the oscillaitng frequency, and V_0 is the translation velocity of the unperturbed flow with respect to the $Ox^{(1)}y^{(1)}$ frame of reference. In the framework of the linearized theory,

$$d\left(x\right)\exp\left(i\omega t\right) = v\left(x^{(1)}, t\right) \tag{13}$$

with v represents the downwash with respect to the new system of coordinates. With respect to the $Ox^{(1)}y^{(1)}$ - system of coordinates general motion is described by the velocity vector field

$$\mathbf{V} = V_0\mathbf{i} + \mathbf{v} \tag{14}$$

where $\mathbf{v}$ represents the perturbation produced by the with into the fluid flow. To evaluate the downwash distribution we employ the general sliding condition

$$\mathbf{V}\cdot\mathbf{n}_{|C^{(1)}} = -\frac{\partial F/\partial t}{|\mathrm{grad}\,F|} \tag{15}$$

In the first approximation, the linearized boundary condition states

$$v\left(x^{(1)}, t\right) = V_0\frac{\partial h^{(1)}}{\partial x^{(1)}}\left(x^{(1)}\right) + i\omega h^{(1)}\left(x^{(1)}\right) \tag{16}$$

We denote into dimensionless function and variable

$$\tilde{h}\left(x\right) = \frac{1}{a}h^{(1)}\left(x^{(1)}\right),\ \tilde{\omega} = \frac{\omega a}{V_0} \tag{17}$$

whence by comparison, Eqs. (13) and (16) yield

$$d\left(x\right) = V_0\left(\frac{\partial h}{\partial x}\left(x\right) + i\tilde{\omega}h\left(x\right)\right) \tag{18}$$

Furthermore, introducing the dimensionless function and variables

$$\tilde{f}\left(x\right) = \frac{1}{V_0^2}f\left(ax\right),\ \tilde{d} = \frac{d}{V_0}$$

Integral equation Eq.(12) becomes

$$\frac{1}{\pi}\int_{-1}^{\prime 1}\tilde{f}(\xi)\exp(-i\tilde{\omega}x_0)\,\mathrm{d}\xi\int_{-\infty}^{*x_0}\frac{\exp(i\tilde{\omega}s)}{s^2}\mathrm{d}s=-2\left(\frac{\mathrm{d}\tilde{h}}{\mathrm{d}x}(x)+i\tilde{\omega}\tilde{h}(x)\right) \tag{19}$$

where $x_0 = x - \xi$. The asterix $*$ indicates the finite part of an integral considered in sense of Hadamard. Eq. (19) represents the two-dimensional integral equation for the pressure jump. The study of it solution will be debated in the followings sections.

3 Possio's Equation

Denote by

$$\mathcal{N}(x_0,\tilde{\omega})=\frac{1}{2\pi}\exp(-i\tilde{\omega}x_0)\,k(x_0,\tilde{\omega}),\; k(x_0,\tilde{\omega})=\int_{-\infty}^{*x_0}\frac{\exp(i\tilde{\omega}s)}{s^2}\mathrm{d}s \tag{20}$$

$$\hat{h}(x)=-2\left(\frac{\mathrm{d}\tilde{h}}{\mathrm{d}x}(x)+i\tilde{\omega}\tilde{h}(x)\right),\quad |x|<1 \tag{21}$$

The integral equation (19) takes the simple form

$$\int_{-1}^{\prime 1}\tilde{f}(\xi)\,\mathcal{N}(x_0,\tilde{\omega})\,\mathrm{d}\xi=\hat{h}(x),\quad |x|<1 \tag{22}$$

known in the literature as *Possio's equation.*

4 On the Kernel Considerations

After one integration, (20) becomes

$$k(x_0,\tilde{\omega})=\frac{\exp(i\tilde{\omega}x_0)}{x_0}+\int_{-\infty}^{\prime x_0}\frac{\exp(i\tilde{\omega}s)\,\mathrm{d}s}{s} \tag{23}$$

Since of fundamental integral relations

$$\int_{-\infty}^{\prime x_0}\frac{\exp(i\tilde{\omega}s)\,\mathrm{d}s}{s}=i\omega\,\mathrm{Ei}(i\tilde{\omega}x_0),\;\mathrm{Ei}(x)=\int_{-\infty}^{\prime x}\frac{\exp(s)}{s}\mathrm{d}s$$

it follows that

$$k(x_0,\tilde{\omega})=\frac{\exp(i\tilde{\omega}x_0)}{x_0}+i\tilde{\omega}\left(\gamma-\ln|x_0|+\int_0^{\tilde{\omega}x_0}\frac{\exp(it)-1}{t}\mathrm{d}t\right). \tag{24}$$

5 Integral Equation of the Thin Profile of Low Frequency.

In the sequel, we consider only wings subjected to oscillations of low frequency ($\tilde{\omega} \ll 1$). This assumption will allow us to find an exact solution on a particular case. Such that, performing an asymptotic expansion with respect to the small parameter $\tilde{\omega}$, and neglecting the terms of $\mathcal{O}(\tilde{\omega}^2)$ order, (24) can be expressed as

$$k(x_0,\tilde{\omega})\sim\frac{\exp(i\tilde{\omega}x_0)}{x_0}-(\ln|x_0|+\Gamma_0)+\mathcal{O}(\tilde{\omega}^2),\;\tilde{\omega}\ll 1 \tag{25}$$

where, we have mentioned before $x_0 = x - \xi$, and $\Gamma_0 = \dfrac{\pi i}{2} - \gamma$.Therefore, by aid with (25), the kernel $\mathcal{N}(x_0, \tilde{\omega})$ becomes

$$\mathcal{N}(x_0, \tilde{\omega}) = \frac{1}{\pi}\frac{1}{x_0} + \frac{\tilde{\omega}}{\pi i}(\ln|x_0| + \Gamma_0) \tag{26}$$

and further, the Possio's equation (22) changes into the final form

$$\frac{1}{\pi}\int_{-1}^{\prime 1} \frac{\tilde{f}(\xi)}{x-\xi} \mathrm{d}\xi + \frac{\tilde{\omega}}{\pi i}\int_{-1}^{\prime 1} \tilde{f}(\xi)(\ln|x-\xi| + \Gamma_0)\,\mathrm{d}\xi = \hat{h}(x), \quad |x| < 1. \tag{27}$$

Eq. (27) represents the integral equation of the low frequency oscillating thin profile. Its solution $\tilde{f}$, yields the pressure jump over the profile, when $\hat{h}(x)$ can be explicited when profile equation is known.

6 Analytical Solution

Solution of the integral equation (27) will be found in two steps.

■ *First.* Consider the equation

$$-\frac{1}{\pi}\int_{-1}^{1} \tilde{f}(\xi)(\ln|x-\xi| + \Gamma_0)\,\mathrm{d}\xi = g(x), \quad |x| < 1. \tag{28}$$

for the unknown function $\tilde{f}$ defined on $(-1, 1)$. Assume that $\tilde{f}$ and g is a hölderian function for $|x| < 1$. We seek to express the general solution and investigate the necessary conditions such that $\tilde{f}$ is vanishing at the TE: $\tilde{f}(1) = 0$. Performing substitutions $(\xi, x) \longmapsto (\theta, \sigma)$ defined by $\xi = \cos\theta$, $x = \cos\sigma$, $(0 \le \sigma \le \pi)$, Eq. (28) becomes

$$-\frac{1}{\pi}\int_{0}^{\prime \pi} F^*(\theta)(\ln|\cos\theta - \cos\sigma| + \Gamma_0)\,\mathrm{d}\theta = G(\sigma), \tag{29}$$

where

$$F^*(\theta) = F(\theta)\sin\theta = \sqrt{1-\xi^2}\,\tilde{f}(\xi)$$

In the Fourier analysis theory one proves that

$$\ln|\cos\theta - \cos\sigma| = -\ln 2 - \sum_{m\ge 1} \frac{2}{m}\cos m\theta \cos m\sigma \tag{30}$$

The function $F^*(\theta)$ within (29) may be prolonged on $(-\pi, 0)$ up to an even function such that be possible an expansion into even trigonometric functions series

$$F^*(\theta) = a_0 + \sum_{n\ge 1} a_n \cos n\theta,\; G(\sigma) = b_0 + \sum_{n\ge 1} b_n \cos n\sigma \tag{31}$$

where, the development coefficients a_i, b_i are defined by

$$a_0 = \frac{1}{\pi}\int_0^{\pi} F^*(\theta)\,\mathrm{d}\theta = \frac{1}{\pi}\int_{-1}^{1} \tilde{f}(\xi)\,\mathrm{d}\xi,\; a_n = \frac{2}{\pi}\int_0^{\pi} F^*(\theta)\cos n\theta\,\mathrm{d}\theta \tag{32}$$

$$b_0 = \frac{1}{\pi}\int_0^{\pi} G(\sigma)\,\mathrm{d}\sigma = \frac{1}{\pi}\int_{-1}^{1} \frac{g(\xi)}{\sqrt{1-\xi^2}}\mathrm{d}\xi,\; b_n = \frac{2}{\pi}\int_0^{\pi} G(\sigma)\cos n\sigma\,\mathrm{d}\sigma \tag{33}$$

whence, by replacing and equating the coefficients, it follows relations

$$b_0 = -a_0\Gamma,\ b_n = \frac{a_n}{n} \qquad (\Gamma = \Gamma_0 - \ln 2) \tag{34}$$

and finaly

$$\frac{1}{\pi}\int_{-1}^{\prime 1} \frac{\sqrt{1-\xi^2}}{x-\xi} g'(\xi)\,\mathrm{d}\xi = -\tilde{f}(x)\sqrt{1-x^2} - \frac{b_0}{\Gamma} \tag{35}$$

Therefore, when $g(x)$ is given, the solution $\tilde{f}(x)$ can be determined from the integral representation

$$\begin{aligned}\tilde{f}(x) &= -\frac{1}{\pi}\sqrt{1-x^2}\int_{-1}^{\prime 1}\frac{g'(\xi)}{\sqrt{1-\xi^2}}\frac{\mathrm{d}\xi}{x-\xi} + \frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{\prime 1}\frac{(x+\xi)\,g'(\xi)}{\sqrt{1-\xi^2}}\mathrm{d}\xi - \\ &\frac{1}{\pi\Gamma\sqrt{1-x^2}}\int_{-1}^{\prime 1}\frac{g(\xi)}{\sqrt{1-\xi^2}}\mathrm{d}\xi\end{aligned} \tag{36}$$

We mention that, from relations (32), (33), and (34) it follows relation

$$\int_{-1}^{1}\tilde{f}(\xi)\,\mathrm{d}\xi = \frac{1}{\Gamma}\int_{-1}^{\prime 1}\frac{g(\xi)}{\sqrt{1-\xi^2}}\mathrm{d}\xi \tag{37}$$

used below at the lift coefficient computation.

A compatibility condition for existence of the solution may be found imposing Kutta-Joukovschi. condition $\tilde{f}(1) = 0$ at TE,

$$\int_{-1}^{\prime 1}\left[(\xi+1)\,g'(\xi) - \frac{g(\xi)}{\Gamma}\right]\frac{\mathrm{d}\xi}{\sqrt{1-\xi^2}} = 0 \tag{38}$$

■ *Second.* We pass to the solving of the equation (27). Assume that $\tilde{f}(x)$ and $\hat{h}(x)$ satisfy Hölder condition on the $(-1,1)$ interval. The existence of the solution is studied in [3]. In order to ensure the uniqueness of the solution, one has to impose the behavior of the unknown $\tilde{f}(\cdot)$ at the end-points of the interval. We are interested to find that solution bounded at the trailing edge, $x = 1$.

Analytical solution is based on reducing of the integral equation to a differential one. Such that, denoting by

$$g(x) = -\frac{1}{\pi}\int_{-1}^{1}\tilde{f}(\xi)\left(\ln|x-\xi| + \Gamma\right)\mathrm{d}\xi \tag{39}$$

and further, Eq. (27) has been reduced to the differential equation

$$g'(x) - i\tilde{\omega}g(x) = \hat{h}(x),\ \ |x| < 1. \tag{40}$$

whose general solution is

$$g(x) = C\exp(i\tilde{\omega}x) + g_0(x),\ g_0(x) = \int_0^x \hat{h}(t)\exp[i\tilde{\omega}(x-t)]\,\mathrm{d}t \tag{41}$$

where C is a constant undetermined *à priori* and $g_0(x)$ is a given function.

Now, Eq. (39) has been reduced to the one of first kind with right hand side given by (41). Its solution may be expressed from (36) and has the general representation

$$\tilde{f}(x) = -\frac{i\tilde{\omega}C}{\pi}\sqrt{1-x^2}T_3 + \frac{C}{\pi\sqrt{1-x^2}}\left(i\tilde{\omega}x + \frac{1}{\Gamma}\right)T_1 + \frac{i\tilde{\omega}C}{\pi\sqrt{1-x^2}}T_2 + \tilde{f}_0(x) \tag{42}$$

where, $\tilde{f}_0(x)$ is acting as a paticular solution and it has been found from(36) when $g(\cdot)$ is changed for $g_0(\cdot)$,

$$\tilde{f}_0(x) = -\frac{1}{\pi}\sqrt{1-x^2}\int_{-1}^{\prime 1}\frac{g_0'(\xi)}{\sqrt{1-\xi^2}}\frac{d\xi}{x-\xi} + \quad (43)$$
$$\frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{\prime 1}\frac{(x+\xi)\, g_0'(\xi)}{\sqrt{1-\xi^2}}d\xi - \frac{1}{\pi\Gamma\sqrt{1-x^2}}\int_{-1}^{\prime 1}\frac{g_0(\xi)}{\sqrt{1-\xi^2}}d\xi$$

with

$$J_0(\tilde{\omega}) = \frac{1}{\pi}\int_0^{\pi}\exp(i\tilde{\omega}\cos\theta)\, d\theta \quad (44)$$

$$J_1(\tilde{\omega}) = -J_0'(\tilde{\omega}) = \frac{1}{\pi}\int_0^{\pi}\cos\theta\exp(i\tilde{\omega}\cos\theta)\, d\theta \quad (45)$$

That is, in terms of Bessel functions, $T1, T2, T3$ can be expressed as follows

$$T_1 = \int_0^{\pi}\exp(i\tilde{\omega}\cos\theta)\, d\theta = \pi J_0(\tilde{\omega}), \quad (46)$$

$$T_2 = \int_0^{\pi}\cos\theta\exp(i\tilde{\omega}\cos\theta)\, d\theta = i\pi J_1(\tilde{\omega}) \quad (47)$$

$$T_3 = \int_0^{\pi}\frac{\exp(i\tilde{\omega}\cos\theta)}{\cos\theta-\cos\sigma}d\theta = \frac{2\pi}{\sqrt{1-x^2}}\sum_{n\geq 1}J_n(\tilde{\omega})\sin n\sigma \quad (48)$$

Finally, replace (46), (47) and (48) into (42), the solution takes a more convenient form. To refine it, should be find the value of the constant C, such that condition $\tilde{f}(1) = 0$ takes place. That is, from (38) and (41), it follows the identity

$$C\left[\left(i\omega + \frac{1}{\Gamma_0}\right)T_1 + i\tilde{\omega}T_2\right] = G_0 \quad (49)$$

whence

$$G_0 = \int_{-1}^{\prime 1}\left[(\xi+1)\, g_0'(\xi) - \frac{1}{\Gamma}g_0(\xi)\right]\frac{d\xi}{\sqrt{1-\xi^2}} \quad (50)$$

Therefore, once (49) and (50) has been evaluated, the pressure jump $\tilde{f}(\cdot)$ may be completely explicited.

7 Aerodynamics Coefficients for the Flat Plate

For our seeking, non-dimensional time will be introduced by $\tilde{t} = \frac{V_0}{a}t$, and for the next considerations, the pressure coefficient

$$C_p\left(x,\tilde{t}\right) = \mathrm{Re}\left\{\tilde{f}(x)\exp(i\tilde{\omega}\tilde{t})\right\} \quad (51)$$

will be used. Among the aerodynamic characteristics of a wing, in the sequel, of a great interest are the lift, moment, and drag coefficients, defined [5] respectively, by the relations

$$C_L\left(\tilde{t}\right) = -2\int_{-1}^{1}C_p\left(ax,\tilde{t}\right)dx, \quad (52)$$

$$C_y\left(\tilde{t}\right) = 2\int_{-1}^{1}xC_p\left(ax, by, \tilde{t}\right)dx \quad (53)$$

$$C_D\left(\tilde{t}\right) = -2\int_{-1}^{1} n_x C_p\left(ax, \tilde{t}\right) \mathrm{d}x, \tag{54}$$

In Eq. (54), n_x represents the normal projection onto Ox - axis and has been set by (2). Evaluation of the average drag coefficient,

$$\widetilde{C}_D = \frac{1}{T}\int_0^T C_D(\tilde{t})\mathrm{d}\tilde{t} \tag{55}$$

where, $T = \dfrac{2\pi}{\tilde{\omega}}$ is oscillation period, represents an important point of our discussion.

8 Analytical Results for the Oscillatory Profiles

Let be $C = C\left(t\right)$ - the wing profile of equation

$$C : F\left(x^{(1)}, y^{(1)}, t\right) \equiv y^{(1)} - h\left(x^{(1)}, t\right) = 0,\ x^{(1)} \in [-a, a] \tag{56}$$

be the wing profile equation with respect to $x^{(1)}Oy^{(1)}$ - system of coordinates related to the profile. For the next partcular cases, the function $h\left(x\right)$ will be specified and aerodynamics coefficients will be evaluated.

■ *Flate Plate.* In this case,

$$\tilde{h}\left(x\right) = -\alpha x,\ n_x = \alpha\exp(i\tilde{\omega}\tilde{t}),\ x \in [-1, 1] \tag{57}$$

and from (21), (57) and (??) it follows

$$\hat{h}\left(x\right) = 2\alpha\left(1 + i\tilde{\omega}x\right), \tag{58}$$

$$g_0\left(x\right) = \frac{4i\alpha}{\tilde{\omega}}\left(1 - \exp\left(i\tilde{\omega}x\right)\right) + 2\alpha x, \tag{59}$$

$$G_0 = 2\pi\alpha\left[2J_0\left(\tilde{\omega}\right) - 1\right] + \pi\alpha i\left[4J_1\left(\tilde{\omega}\right) - \frac{\tilde{\omega}}{\Gamma}\right],$$

From (??) and (??) the function $g\left(x\right)$ may be expressed as

$$g\left(x\right) = \alpha\left(C_0 - \frac{4i}{\tilde{\omega}}\right)\exp\left(i\tilde{\omega}x\right) - \alpha\left(2x + \frac{4i}{\tilde{\omega}}\right) \tag{60}$$

for the next considerations we have set $C = \alpha C_0$, the constant C being evaluated from (??), (??). For the lift coefficient, one applies the relating formula (37) such that

$$\int_{-1}^{1}\tilde{f}\left(x\right)\mathrm{d}x = \frac{\pi\alpha}{\Gamma_0}\left(\frac{1}{2}\tilde{\omega}i + C_0 J_0\left(\tilde{\omega}\right)\right) \tag{61}$$

For the lift coefficient we obtain the formula

$$C_L\left(\tilde{t}\right) = -\pi\alpha\,\mathrm{Re}\left\{\frac{1}{\Gamma_0}\left(\tilde{\omega}i + 2C_0 J_0\left(\tilde{\omega}\right)\right)\exp(i\tilde{\omega}\tilde{t})\right\} \tag{62}$$

Further computations, needs to express the solution

$$\begin{aligned}\tilde{f}\left(x\right) \quad = \quad & -2\alpha\left(4 + i\tilde{\omega}C_0\right)\sum_{n\geq 1} J_n\left(\tilde{\omega}\right)\sin n\sigma + \frac{\alpha}{\sqrt{1-x^2}}\left(4i - \tilde{\omega}C_0\right)J_1\left(\tilde{\omega}\right) - \\ & \frac{x}{\sqrt{1-x^2}}\left[-2 + \left(4i + \tilde{\omega}C_0\right)J_0\left(\tilde{\omega}\right)\right] - \frac{\alpha}{\Gamma\sqrt{1-x^2}}\left(C_0 J_0\left(\tilde{\omega}\right) + \frac{1}{2}i\tilde{\omega}\right)\end{aligned} \tag{63}$$

Therefore, from integral

$$\int_{-1}^{1} x\ \tilde{f}(x)\,\mathrm{d}x = -\frac{\pi\alpha}{2}(4 + i\tilde{\omega}C_0)(2\alpha - J_0(\tilde{\omega})) \tag{64}$$

the moment coefficient has been found as

$$C_y(\tilde{t}) = -\pi\alpha\,\mathrm{Re}\left\{(4 + i\tilde{\omega}C_0)(2\alpha - J_0(\tilde{\omega}))\exp(i\tilde{\omega}\tilde{t})\right\} \tag{65}$$

The drag coefficient may be obtained from (54) as

$$C_D(\tilde{t}) = -\pi\alpha^2\cos\tilde{\omega}\tilde{t}\,\mathrm{Re}\left\{\frac{1}{\Gamma_0}(\tilde{\omega}i + 2C_0J_0(\tilde{\omega}))\exp(i\tilde{\omega}\tilde{t})\right\} \tag{66}$$

The average drag coefficient

$$\tilde{C}_D = -\frac{\pi\alpha^2}{4}\phi(\tilde{\omega})$$

where funtion $\phi(\tilde{\omega}) \geq 0$ for some $\tilde{\omega} > \tilde{\omega}_{cr} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

9 Analytical Results for the Undulatory Profile

In the sequel, one consider those profiles of equation (56) for which,

$$h\left(x^{(1)}\right) = -\alpha\exp\left(-i\omega_1 x^{(1)}\right),\ x \in [-a, a] \tag{67}$$

Eq. (2) yields the projection of normal vector on the Ox - axis,

$$n_x = i\alpha\omega_1\exp\left(i\tilde{\omega}\tilde{t} - i\omega_1 x\right),\ \omega_1 = a\tilde{\omega},\ x \in [-1, 1] \tag{68}$$

and from (21) and (67) it follows

$$\hat{h}(x) = 2i\alpha(\tilde{\omega} - \omega_1)\exp(-i\omega_1 x) \tag{69}$$

$$g_0(x) = \frac{2\alpha(\tilde{\omega} - \omega_1)}{\tilde{\omega} + \omega_1}\left[\exp\left(i\tilde{\omega}\tilde{t}\right) - \exp(-i\omega_1 x)\right] \tag{70}$$

$$g(x) = \alpha C_0\exp(i\tilde{\omega}x) + \frac{2i\alpha(\tilde{\omega} - \omega_1)}{\tilde{\omega} + \omega_1}\left[\exp\left(i\tilde{\omega}\tilde{t}\right) - \exp(-i\omega_1 x)\right]$$

$$\int_{-1}^{1}\tilde{f}(x)\,\mathrm{d}x = \frac{\pi\alpha C_0}{\Gamma}J_0(\tilde{\omega}) + \frac{2\pi\alpha(\tilde{\omega} - \omega_1)}{\tilde{\omega} + \omega_1}(J_0(\tilde{\omega}) + J_0(\omega_1))$$

Then, (50) and (70) it follows

$$G_0 = -\frac{2\pi\alpha(\tilde{\omega} - \omega_1)}{(\tilde{\omega} + \omega_1)}\Big[\tilde{\omega}(J_1(\tilde{\omega}) - iJ_0(\omega_1)) + \omega_1(J_1(\omega_1) + iJ_0(\omega_1)) - \frac{1}{\Gamma_0}(J_0(\tilde{\omega}) - iJ_0(\omega_1))\Big] \tag{71}$$

The constant, C, can be evaluated from (50), (46), and (71),

$$C_0 = \frac{2\alpha(\tilde{\omega} - \omega_1)}{(\tilde{\omega} + \omega_1)\left[\tilde{\omega}i_1(\tilde{\omega}) + \left(\tilde{\omega} + \frac{i}{\Gamma_0}\right)J_0(\tilde{\omega})\right]}\Big[\tilde{\omega}(J_1(\tilde{\omega}) - iJ_0(\omega_1)) + \omega_1(J_1(\omega_1) + iJ_0(\omega_1)) - \frac{1}{\Gamma_0}(J_0(\tilde{\omega}) - iJ_0(\omega_1))\Big] \tag{72}$$

From (??) and (70) one obtains the function $g(x)$ may be expressed as

$$g(x) = C\exp(i\tilde{\omega}x) - 2\alpha x - \frac{4\alpha i(\tilde{\omega}-\omega_1)}{\tilde{\omega}+\omega_1}(\exp(i\tilde{\omega}x) - \exp(-i\omega_1 x)) \tag{73}$$

For the lift coefficient, we get the evaluation

$$C_L(\tilde{t}) = 2\pi\alpha\,\mathrm{Re}\{[2J_0(\omega_1)(\tilde{\omega}-\omega_1) - J_0(\tilde{\omega})[(2+C_0)\tilde{\omega} - (2-C_0)\omega_1]\frac{\exp(i\tilde{\omega}\tilde{t})}{\Gamma_0}\right\} \tag{74}$$

For the moment coefficient we will proceed as above, and first determine the pressure jump for this case.

$$\tilde{f}(x) = \frac{2\alpha}{\Gamma_0(\tilde{\omega}+\omega_1)\sqrt{1-x^2}}\{J_0(\omega_1)(\tilde{\omega}-\omega_1) - J_0(\tilde{\omega})\times [(2+C_0)\tilde{\omega} - (2-C_0)\omega_1]\} - 2i\alpha\tilde{\omega}C\sum_{n\geq 1}J_n(\omega)\sin n\sigma - \frac{8i\alpha(\tilde{\omega}-\omega_1)}{(\tilde{\omega}+\omega_1)}\left(\tilde{\omega}\sum_{n\geq 1}J_n(\omega)\sin n\sigma + \omega_1\sum_{n\geq 1}J_n(\omega_1)\sin n\sigma\right) + \frac{2\alpha\omega_1(\tilde{\omega}-\omega_1)}{(\tilde{\omega}+\omega_1)\sqrt{1-x^2}}[ixJ_0(\omega_1) + J_1(\omega_1)] - \frac{\alpha\tilde{\omega}}{(\tilde{\omega}+\omega_1)\sqrt{1-x^2}}[(2+C_0)\tilde{\omega} - (2-C_0)\omega_1][-ixJ_0(\tilde{\omega}) + J_1(\tilde{\omega})] \tag{75}$$

whence the moment coefficient will be

$$C_y(\tilde{t}) = 2\pi\alpha\,\mathrm{Re}\left\{\left[4\tilde{\omega}C_0\sum_{n\geq 1}J_n(\omega)\sin n\sigma - \frac{8(\tilde{\omega}-\omega_1)}{(\tilde{\omega}+\omega_1)}\times \times\left(\tilde{\omega}\sum_{n\geq 1}J_n(i\omega)\sin n\sigma + \omega_1\sum_{n\geq 1}J_n(\omega_1)\sin n\sigma\right) + \frac{\pi(\tilde{\omega}-\omega_1)}{(\tilde{\omega}+\omega_1)}\times \times[\omega_1(\tilde{\omega}-\omega_1)J_0(\omega_1) + \frac{\tilde{\omega}}{2}[(2+C_0)\tilde{\omega} - (2-C_0)\omega_1]J_0(\tilde{\omega})\right] i\exp(i\tilde{\omega}\tilde{t})\right\} \tag{76}$$

For the drag coefficient computation one takes into account that

$$\int_{-1}^{1}\frac{x\sin(\omega_1 x)}{\sqrt{1-x^2}}\mathrm{d}x = \pi J_1(\omega_1)$$

so that, with (54) one obtains

$$C_D(\tilde{t}) = -2\pi\alpha^2\frac{\omega_1 J_1(\omega_1)}{\tilde{\omega}+\omega_1}\mathrm{Re}\{[(\tilde{\omega}(2+C_0)\tilde{\omega} - (2-C_0)\omega_1)J_0(\tilde{\omega}) + 2(\tilde{\omega}-\omega_1)J_0(\omega_1)]i\exp(2i\tilde{\omega}\tilde{t})\} \tag{77}$$

The average drag coefficient

$$\tilde{C}_D = -\frac{\pi\alpha^2}{4}\phi_1(\tilde{\omega})$$

where funtion $\phi_1(\tilde{\omega}) \geq 0$ for some $\tilde{\omega} > \tilde{\omega}_{cr} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

10 Self-Propulsion

Notice that when $\tilde{\omega}$ is such that $\tilde{C}_D < 0$, then a propulsive force is appearing.

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