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SOME QUADRATURE FORMULAS BASED ON LINEAR AND POSITIVE OPERATORS

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Abstract: *The paper is a survey concerning quadrature formulas obtained by using linear and positive operators. We recall here the Bernstein quadrature formula, the Schurer quadrature formula, the Stancu quadrature formula and the Schurer-Stancu quadrature formula. We present also some new results regarding the composite quadrature formula of Bernstein type.*

Keywords: *Bernstein operator, Schurer operator, Stancu operator, Schurer-Stancu operator, Bernstein quadrature formula, Schurer quadrature formula, Stancu quadrature formula, Schurer-Stancu quadrature formula, composite quadrature formula*
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1 The linear and positive operators used

In this section we recall the linear and positive operators used for the construction of the quadrature formulas which we shall present.
First, let us to denote $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Next, let $p \in \mathbb{N}_0$ be given and let α, β be real parameters satisfying the condition $0 \le \alpha \le \beta$.
The Schurer-Stancu's operator is defined [2] for any $n \in \mathbb{N}$, any $f \in C([0, 1+p])$ and any $x \in [0, 1]$ by:

$$\left(\tilde{S}_{n,p}^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{n+p} \tilde{p}_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right) \tag{1}$$

where

$$\tilde{p}_{n,k}(x) = \binom{n+p}{k} x^k (1-x)^{n+p-k} \tag{2}$$

are the fundamental Schurer's polynomials.
Many approximation properties of the operator (1) can be found in the monograph [4] and the references therein.
Note that for $\alpha = \beta = 0$ and $p \in \mathbb{N}_0$, the operator (1) is the Schurer operator [9] defined by:

$$\left(\tilde{B}_{n,p} f\right)(x) = \sum_{k=0}^{n+p} \tilde{p}_{n,k}(x) f\left(\frac{k}{n}\right). \tag{3}$$

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For $p = 0$ the operator (1) reduces to the Stancu's operator [12], defined by:

$$\left(S_n^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right) \tag{4}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \tag{5}$$

are the fundamental Bernstein's polynomials.
For $\alpha = \beta = 0$ and $p = 0$, one arrives to the classical Bernstein operator [7] defined by:

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right). \tag{6}$$

Using the operator (1), for any $f \in C([0, 1 + p])$ one arrives to the following approximation formula:

$$f = \tilde{S}_{n,p}^{(\alpha,\beta)} f + \tilde{R}_{n,p}^{(\alpha,\beta)} f \tag{7}$$

where $\tilde{R}_{n,p}^{(\alpha,\beta)} f$ denotes the remainder term.
Following the Stancu's ideas [10], [11], [12], [13], in [4] was proved:

Theorem 1.1. *For any $f \in C([0, 1 + p])$ and any $x \in [0, 1]$, the remainder term of (7) can be represented under the form:*

$$\begin{aligned}\left(\tilde{R}_{n,p}^{(\alpha,\beta)} f\right)(x) =& \frac{1}{n+\beta}\{(\beta - p)x - \alpha\}[\xi_1, \xi_2; f] \\ &- \frac{n+p}{(n+\beta)^2} x(1-x)[\eta_1, \eta_2, \eta_3; f]\end{aligned} \tag{8}$$

where the brackets denote divided differences and $0 \le \xi_1 < \xi_2 \le 1$, $0 \le \eta_1 < \eta_2 < \eta_3 \le 1$.

Corollary 1.1. *Let $f \in C([0, 1+p])$ be two times differentiable on $[0, 1]$ having the derivatives f' and f'' finite on $[0, 1]$. Then, for any $x \in [0, 1]$ there exist $\xi, \eta \in [0, 1]$ such that:*

$$\left(\tilde{R}_{n,p}^{(\alpha,\beta)} f\right)(x) = \frac{1}{n+\beta}\{(\beta - p)x - \alpha\} f'(\xi) - \frac{n+p}{2(n+\beta)^2} x(1-x) f''(\eta). \tag{9}$$

Let $\alpha = \beta = 0$ and $p \in N_0$. Then, the approximation formula (7) becomes the following Schurer's approximation formula:

$$f = \tilde{B}_{n,p} f + \tilde{R}_{n,p} f. \tag{10}$$

As a consequence of Theorem 1.1. and Corollary 1.1., it follows:

Corollary 1.2. *i) For any $f \in C([0, 1 + p])$ and any $x \in [0, 1]$ the remainder term of (10) can be represented under the form:*

$$\left(\tilde{R}_{n,p} f\right)(x) = -\frac{px}{n}[\xi_1, \xi_2; f] - \frac{n+p}{n^2} x(1-x)[\eta_1, \eta_2, \eta_3; f] \tag{11}$$

where $0 \le \xi_1 < \xi_2 \le 1$ and $0 \le \eta_1 < \eta_2 < \eta_3 \le 1$.

ii) Let $f \in C([0, 1 + p])$ be two times differentiable on $[0, 1]$ having the derivatives f' and f'' finite on $[0, 1]$. Then, for any $x \in [0, 1]$ there exist $\xi, \eta \in [0, 1]$ such that:

$$\left(\tilde{R}_{n,p} f\right)(x) = -\frac{px}{n} f'(\xi) - \frac{n+p}{2n^2} x(1-x) f''(\eta). \tag{12}$$

For $p = 0$, the approximation formula (7) becomes the Stancu's approximation formula:

$$f = S_n^{(\alpha,\beta)} f + R_n^{(\alpha,\beta)} f. \tag{13}$$

As a consequence of Theorem 1.1. and Corollary 1.1., it follows:

Corollary 1.3. *i) For any $f \in C([0,1])$ and any $x \in [0,1]$ the remainder term of Stancu's approximation formula (13) can be expressed under the form:*

$$\left(R_n^{(\alpha,\beta)} f\right)(x) = \frac{1}{n+\beta}\{\beta x - \alpha\}[\xi_1, \xi_2; f] - \frac{n}{(n+\beta)^2} x(1-x)[\eta_1, \eta_2, \eta_3; f], \tag{14}$$

where $0 \le \xi_1 < \xi_2 \le 1$ and $0 \le \eta_1 < \eta_2 < \eta_3 \le 1$.

ii) Let $f \in C([0,1])$ be two times differentiable on $[0,1]$ having the derivatives f' and f'' finite on $[0,1]$. Then, for any $x \in [0,1]$ there exist $\xi, \eta \in [0,1]$ such that:

$$\left(R_n^{(\alpha,\beta)} f\right)(x) = \frac{1}{n+\beta}\{\beta x - \alpha\} f'(\xi) - \frac{n}{2(n+\beta)^2} x(1-x) f''(\eta). \tag{15}$$

Suppose that $\alpha = \beta = 0$ and $p = 0$. In this case we get the classical Bernstein's approximation formula:

$$f = B_n f + R_n f. \tag{16}$$

As a consequence of Theorem 1.1. and Corollary 1.1., for approximation formula (16) we get the result obtained by Tiberiu Popoviciu contained in:

Corollary 1.4. *[8]*

i) For any $f \in C([0,1])$ and any $x \in [0,1]$, the remainder term of (16) can be represented under the form:

$$\left(R_n f\right)(x) = -\frac{1}{n} x(1-x)[\xi_1, \xi_2, \xi_3; f], \tag{17}$$

where $0 \le \xi_1 < \xi_2 < \xi_3 \le 1$.
It is well known the following estimation of the remainder term of (16), [16].

ii) Let $f \in C([0,1])$ be two times differentiable on $[0,1]$ having the derivatives f' and f'' finite on $[0,1]$. Then, for any $x \in [0,1]$ there exists $\xi \in [0,1]$ such that, the following inequality:

$$\left|\left(R_n f\right)(x)\right| \le \frac{1}{2n} x(1-x) M_2[f] \tag{18}$$

holds, where:

$$M_2[f] = \max_{x \in [0,1]} |f''(x)|. \tag{19}$$

2 quadrature formulas based on the operator $\tilde{S}_{n,p}^{(\alpha,\beta)}$

Let $f \in C([0, 1+p])$ be given and let

$$\int_0^1 f(x)dx = \sum_{k=0}^{n+p} A_{n+p,k}^{(\alpha,\beta)} f\left(\frac{k+\alpha}{n+\beta}\right) + r_{n,p}^{(\alpha,\beta)}(f) \tag{20}$$

be the Schurer-Stancu type quadrature formula [3], where the coefficients are expressed by:

$$A_{n+p,k}^{(\alpha,\beta)} = \frac{1}{n+p+1} \tag{21}$$

for any $k = \overline{0, n+p}$.

Lemma 2.1. *[5] Let $e_j(x) = x^j$ (j non-negative integer) be the test monomials. The following identities:*

$$r_{n,p}^{(\alpha,\beta)}(e_0) = 0; \tag{22}$$

$$r_{n,p}^{(\alpha,\beta)}(e_1) = \frac{\beta - 2\alpha - p}{2(n+\beta)}; \tag{23}$$

$$r_{n,p}^{(\alpha,\beta)}(e_2) = \frac{2(\beta - p)(2n+\beta+p) - (n+p)(6\alpha+1) - 6\alpha^2}{6(n+\beta)^2}, \tag{24}$$

hold.

Remark 2.1. *i) In general $r_{n,p}^{(\alpha,\beta)}(e_1) \neq 0$, i. e., the degree of exactness of (20) is 0.*

ii) For $\beta = 2\alpha + p$, yield $r_{n,p}^{(\alpha,2\alpha+p)}(e_1) = 0$, $r_{n,p}^{(\alpha,2\alpha+p)}(e_2) \neq 0$, i. e., (20) has the degree of exactness 1.

In what follows, we are dealing with the Schurer-Stancu quadrature formula having the degree of exactness 1, i. e., with the following quadrature formula:

$$\int_0^1 f(x)dx = \frac{1}{n+p+1}\sum_{k=0}^{n+p} f\left(\frac{k+\alpha}{n+p+2\alpha}\right) + r_{n,p}^{(\alpha,2\alpha+p)}(f). \tag{25}$$

Theorem 2.1. *[5] If*

i) $f \in C([0,1+p]) \cap C^2([0,1])$;

ii) $\beta = 2\alpha + p$ and $n + p > 4\alpha^2$, the remainder term of Schurer-Stancu quadrature formula (25) can be represented under the form:

$$r_{n,p}^{(\alpha,2\alpha+p)}(f) = \frac{(2\alpha-1)n + 2\alpha^2 + (2\alpha-1)p}{12(n+p+2\alpha)^2} f''(\xi), \tag{26}$$

where $\xi \in]0,1[$.

Remark 2.2. *The minimum value of the remainder term of the quadrature formula (25) is obtained for $\alpha = \frac{1}{2}$.*

Theorem 2.2. *[5] If*

i) $f \in C([0,1+p]) \cap C^2([0,1])$;

ii) $n + p > 4\alpha^2$, the optimal quadrature formula of Schurer-Stancu type is the following:

$$\int_0^1 f(x)dx = \frac{1}{n+p+1}\sum_{k=0}^{n+p} f\left(\frac{2k+1}{2n+2p+2}\right) + \frac{1}{24(n+p+1)^2} f''(\xi). \tag{27}$$

Remark 2.3. *i) For $\alpha = \beta = 0$ and $p \neq 0$, form (20) we get the Schurer quadrature formula:*

$$\int_0^1 f(x)dx = \frac{1}{n+p+1}\sum_{k=0}^{n+p} f\left(\frac{k}{n}\right) + r_{n,p}(f). \tag{28}$$

ii) For $p = 0$, from (20) we get the Stancu quadrature formula:

$$\int_0^1 f(x)dx = \frac{1}{n+1}\sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right) + r_n^{(\alpha,\beta)}(f). \tag{29}$$

iii) For $p = 0$, from (27) we get the Stancu's optimal quadrature formula:

$$\int_0^1 f(x)dx = \frac{1}{n+1}\sum_{k=0}^{n} f\left(\frac{2k+1}{2n+2}\right) + \frac{1}{24(n+1)^2} f''(\xi). \tag{30}$$

3 The composite Bernstein type quadrature formula

Starting with the Bernstein approximation formula (16) in [16] the following Bernstein quadrature formula:

$$\int_0^1 f(x)dx = \sum_{j=0}^{n} A_j f\left(\frac{j}{n}\right) + R_n[f] \tag{31}$$

is obtained, where

$$A_j = \frac{1}{n+1}, \ (\forall)\ j = \overline{0,n} \tag{32}$$

and

$$|R_n[f]| \le \frac{1}{12n} M_2[f]. \tag{33}$$

The focus of this section is to construct the composite Bernstein type quadrature formula. For this aim, the interval $[0,1]$ will be divided in m equally spaced subintervals $\left[\frac{k-1}{m}, \frac{k}{m}\right]$, $k = \overline{1,m}$. Each interval at his turn will be divided in n_k, $k = \overline{1,m}$ subintervals. On each such type of interval $\left[\frac{k-1}{m}, \frac{k}{m}\right]$, $k = \overline{1,m}$, with corresponding knots $n_k + 1$, $k = \overline{1,m}$, the Bernstein quadrature formula (31) will be applied. Next, adding the mentioned quadrature formulas the desired Bernstein type quadrature formula on interval $[0,1]$ will be obtained.
Let $[a,b]$ be real interval, which in n_p equally spaced subintervals will be divided.
We start with two auxilliary results.

Lemma 3.1. *Suppose that $a, b \in \mathbb{R}$, $a < b$ and $f \in C([a,b])$. Then, the Bernstein polynomial associated to the function f is defined by:*

$$(B_{n_p} f)(x) = \frac{1}{(b-a)^{n_p}} \sum_{j=0}^{n_p} \binom{n_p}{j} (x-a)^j (b-x)^{n_p - j} f\left(a + j\frac{b-a}{n_p}\right). \tag{34}$$

Proof. It is easy to observe that the correspondence $t \to \frac{x-a}{b-a}$ transform the interval $[a,b]$ in the interval $[0,1]$. Taking (5), (6) and the above remark into account, yields (34). □

Lemma 3.2. *Suppose that $a, b \in \mathbb{R}$, $a < b$ and $f \in C^2([a,b])$. Then, the remainder term of the Bernstein approximation formula on $[a,b]$ verifies the inequality:*

$$|(R_{n_p} f)(x)| \le \frac{(x-a)(b-x)}{2n_p(b-a)^2} M_2[f] \tag{35}$$

where $M_2[f] = \max_{x \in [a,b]} |f''(x)|$.

Proof. One applies (18), taking the transformation $t \to \frac{x-a}{b-a}$ into account. □

In what follows, let us to consider the interval $[0,1]$ divided in the equally spaced subintervals $\left[\frac{k-1}{m}, \frac{k}{m}\right]$, $k = \overline{1,m}$. Each interval at his turn will be divided in n_k, $k = \overline{1,m}$ subintervals. In each interval $\left[\frac{k-1}{m}, \frac{k}{m}\right]$ one considers the distinct knots $x_j = \frac{kn_k - n_k + j}{mn_k}$, $j = \overline{0, n_k}$. Applying Lemma 3.1., yields the following Bernstein type polynomial:

$$(B_{n_k} f)(x) = m^{n_k} \sum_{j=0}^{n_k} \binom{n_k}{j} \left(x - \frac{k-1}{m}\right)^j \left(\frac{k}{m} - x\right)^{n_k - j} f\left(\frac{kn_k - n_k + j}{mn_k}\right). \tag{36}$$

The corresponding Bernstein type approximation formula on the interval $\left[\frac{k-1}{m}, \frac{k}{m}\right]$ becomes:

$$f = B_{n_k} f + R_{n_k} f. \tag{37}$$

If $f \in C^2([0,1])$, the remainder term of (37) verifies the inequality:

$$|(R_{n_k} f)(x)| \leq \frac{\left(x - \frac{k-1}{m}\right)\left(\frac{k}{m} - x\right)}{2n_k} m^2 M_2[f] \tag{38}$$

where $M_2[f] = \max_{x \in [a,b]} |f''(x)|$.

Theorem 3.1. *If $f \in C^2([0,1])$, the following Bernstein type quadrature formula*

$$\int_{\frac{k-1}{m}}^{\frac{k}{m}} f(x)dx = \sum_{j=0}^{n_k} A'_j f\left(\frac{kn_k - n_k + j}{mn_k}\right) + R'_{n_k}[f] \tag{39}$$

holds, for any $k = \overline{1,m}$, where

$$A'_j = \frac{1}{m(n_k+1)}, \ (\forall)\ j = \overline{0, n_k} \tag{40}$$

and

$$|R'_{n_k}[f]| \leq \frac{1}{12mn_k} M_2[f]. \tag{41}$$

Proof. Integrating (37) on $\left[\frac{k-1}{m}, \frac{k}{m}\right]$ and taking (36) into account, yields:

$$\begin{aligned} A'_j &= m^{n_k} \binom{n_k}{j} \int_{\frac{k-1}{m}}^{\frac{k}{m}} \left(x - \frac{k-1}{m}\right)^j \left(\frac{k}{m} - x\right)^{n_k - j} dx \\ &= m^{n_k} \binom{n_k}{j} \int_0^1 \left(\frac{t}{m}\right)^j \left[\frac{1}{m}(1-t)\right]^{n_k - j} \frac{1}{m} dt \\ &= \frac{1}{m} \binom{n_k}{j} \int_0^1 t^j (1-t)^{n_k - j} dt. \end{aligned}$$

The last integral is the Euler function of first kind $B(j+1, n_k - j + 1)$. Taking the well known properties of Euler function of first kind into account, it follows:

$$A'_j = \frac{1}{m}\binom{n_k}{j} B(j+1, n_k - j + 1) = \frac{1}{m} \frac{n_k!}{j!(n_k - j)!} \frac{j!(n_k - j)!}{(n_k+1)!} = \frac{1}{m(n_k+1)}.$$

For the remainder term, taking (38) into account, we get:

$$|R'_{n_k}[f]| \leq M_2[f] \frac{m^2}{2n_k} \int_{\frac{k-1}{m}}^{\frac{k}{m}} \left(x - \frac{k-1}{m}\right)\left(\frac{k}{m} - x\right) dx. \tag{42}$$

But $\int_{\frac{k-1}{m}}^{\frac{k}{m}} \left(x - \frac{k-1}{m}\right)\left(\frac{k}{m} - x\right) dx = \frac{1}{6m^3}$ and from (42) one arrives to the desired inequality (41). □

Theorem 3.2. *For any $f \in C^2[0,1]$, the following composite Bernstein type quadrature formula*

$$\int_0^1 f(x)dx = \frac{1}{m}\sum_{k=1}^{m}\sum_{j=0}^{n_k}\frac{1}{n_k+1}f\left(\frac{kn_k - n_k + j}{mn_k}\right) + R_{n_m}[f] \tag{43}$$

holds, where

$$|R_{n_m}[f]| \leq \frac{1}{12m}M_2[f]\sum_{k=1}^{m}\frac{1}{n_k}. \tag{44}$$

Proof. Adding the Bernstein type quadrature formulas (39) for $k = \overline{1,m}$, we get the following composite Bernstein type quadrature formula:

$$\int_0^1 f(x)dx = \frac{1}{m}\sum_{k=1}^{m}\sum_{j=0}^{n_k} f\left(\frac{kn_k - n_k + j}{mn_k}\right) + \sum_{k=1}^{m} R'_{n_k}[f]. \tag{45}$$

Denoting $R_{n_m}[f] = \sum\limits_{k=1}^{m} R'_{n_k}[f]$ and taking (41) into account, yields:

$$|R_{n_m}[f]| \leq \sum_{k=1}^{m}|R'_{n_k}[f]| \leq \frac{1}{12m}M_2[f]\sum_{k=1}^{m}\frac{1}{n_k}. \tag{46}$$

□

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