THE USING OF THE INTEGRATING FACTOR IN SOLVING OF DIFFERENTIAL STOCHASTIC EQUATION

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Abstract: *More and more random phenomena in our environment are modulated by differential stochastic equations. The problem is finding of techniques solving differential stochastic equation using similar ways to the one we use for solving classic differential calculus. One such way is solving differential stochastic equation with the integrating factor. In the following paper is exemplifies of this solving way in respect of the Itô's rules of the stochastic differential calculus.*

Keywords: differential stochastic equations, Ito's rules, integrating factor

1 Introduction

The technique of the integrating factor in solving of stochastic differential equation is to multiply the both terms with same factor that confinable groped we roll of stochastic differential expression.

Let beB_t is the Brownian 1-dimensional motion, $B_0 = 0$ we remain the same rules specifies the stochastic differential Itô, [4]:

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \ \ dB_t \cdot dB_t = dt. \tag{1}$$

2 Main Results

Let be x_t 1-dimensional stochastic process,B_t the Brownian 1-dimensional motion which satisfies the following stochastic differential equation:

$$dx_t = x_t \cdot dt + dB_t \tag{2}$$

Statement 1: The solution of the equation (1) is the following stochastic process:

$$x_t = B_t - B_0 \cdot e^t + e^t \cdot \int_0^t B_s \cdot e^{-s} ds \tag{3}$$

Where, the integral of the right side is the It integral, [5].

P*roof* : We use the integrating factor, and multiplies the both sides of equation (2) with e^{-t}, and obtain:

$$e^{-t} \cdot dx_t = e^{-t} \cdot x_t \cdot dt + e^{-t} \cdot dB_t \tag{4}$$

To regrouping the terms of the equation (4), we rewrite:
$e^{-t} \cdot dx_t - e^{-t} \cdot x_t \cdot dt = e^{-t} \cdot B_t$. (4')
To find the stochastic differential of the product $e^{-t} x_t$ process:

Paper presented at The VI-th International Conference on Nolinear Analysis and Applied Mathematics (ICNAAM), Târgovişte, 21-22 nov, 2008

$$d\left(e^{-t}\cdot x_t\right)=x_t\cdot d\left(e^{-t}\right)+e^{-t}\cdot dx_t+d\left(e^{-t}\right)\cdot dx_t \tag{5}$$

Because, $de^{-t}=-e^{-t}\cdot dt$, we find:
$d\left(e^{-t}\cdot x_t\right)=-e^{-t}\cdot x_t\cdot dt+e^{-t}\cdot dx_t-e^{-t}\cdot dt\cdot dx_t$ (5')

Now, we use equations (1) and (2) by introduction them in the right side of (5') equations, it follows that:
$d\left(e^{-t}\cdot x_t\right)=e^{-t}\cdot dx_t-e^{-t}\cdot x_t\cdot dt$ (5")

To compare this equation with equation (4') we deduct:

$$d\left(e^{-t}\cdot x_t\right)=e^{-t}\cdot dB_t \tag{6}$$

Therefore, we obtain:
$e^{-t}\cdot x_t=\int_0^t e^{-s}\cdot dB_s$ (6')

The integral of right side is Itô integral, and equation (6") is immediately.
$x_t=e^{t}\cdot\int_0^t e^{-s}\cdot dB_s$ (6")

We apply now the integrating by parts for It integral to find the following form for equation (6"):

$$x_t=e^{t}\left[e^{-t}\cdot B_t-B_0-\int_0^t B_s\cdot e^{-s}\cdot ds\right] \tag{7}$$

Or, therefore result:
$x_t=B_t-B_0\cdot e^{t}+e^{t}\cdot\int_0^t B_s\cdot e^{-s}ds$ (7')

Finally, for $B_0=0$ the forma of solution of stochastic differential equation (2) is:

$$x_t=B_t+e^{t}\cdot\int_0^t B_s\cdot e^{-s}ds \tag{8}$$

Statement 2: If the process x_t is the solution of equation (2), and B_t is the Brownian 1-dimensional motion then the expectation value and the variation of the process are:

$$E\left[x_t\right]=-B_0\cdot e^{t},\ \ Var\left[x_t\right]=t+B_0^2\cdot e^{2t} \tag{9}$$

For case$B_0=0$:
$E\left[x_t\right]=0,\ \ Var\left[x_t\right]=t$ (9')

Proof: To evaluate the expectation value and the variation of stochastic process (7') using their properties, [8]:

$$E\left[x_t\right]=E\left[B_t\right]-E\left[B_0\cdot e^{t}\right]+E\left[e^{t}\cdot\int_0^t e^{-s}B_s\cdot ds\right] \tag{10}$$

Because
$E\left[B_t\right]=0,\ \ E\left[B_0\cdot e^{t}\right]=B_0\cdot e^{t},\ \ E\left[e^{t}\cdot\int_0^t e^{-s}\cdot B_s\cdot ds\right]=e^{t}\int_0^t e^{-s}\cdot E\left[B_s\right]\cdot ds=0, E\left[B_t^2\right]=t$ (10')

We have (9). To evaluate the variation for the case$B_0=0$, we have:

$$Var\left[x_t\right]=E\left[\left(x_t-E\left[x_t\right]\right)^2\right]=E\left[x_t^2\right] \tag{11}$$

But
$x_t^2=B_t^2+2\cdot B_t\cdot e^{t}\cdot\int_0^t B_s\cdot e^{-s}ds+\left[e^{t}\cdot\int_0^t B_s\cdot e^{-s}ds\right]^2$, (11')

With the properties of expectation value of Brownian motion, [8], similar (10'), we obtain the form of (9').

For the case $B_0\neq 0$, the relation (11') is more complicated, but the important term is the one which contains the $B_0^2\cdot e^{2t}$, because his expectation value is not equal to zero, all the other terms have their expectation value equal to zero, and then we get the form of (9) for the variation stochastic process.

Statement 3: Generalization. The Ornstein-Uhlenbeck or Langevin equation. Let be the following stochastic differential equation:

$$dx_t = \mu \cdot x_t \cdot dt + \sigma \cdot dB_t \tag{12}$$

where μ, σ are real constants, $B_t \in R$, is the Brownian motion.

The solution of (12) is a stochastic process with form:

$$x_t = \sigma \left(B_t - B_0 \cdot e^{\mu \cdot t}\right) + \mu \cdot \sigma \cdot e^{\mu \cdot t} \cdot \int_0^t B_s \cdot e^{-\mu \cdot s} \cdot ds \tag{13}$$

The expectation value and the variation of this process are:

$$E\left[x_t\right] = -\sigma \cdot B_0 \cdot e^{\mu \cdot t}, Var\left[x_t\right] = \sigma^2 \cdot t + \sigma^2 \cdot B_0^2 \cdot e^{2\mu \cdot t} \tag{14}$$

The process (13) is called the Ornstein-Uhlenbeck process.

Proof: The prove is similar, we use the integrating factor, $e^{-\mu \cdot t}$, and compare with the stochastic differential, $d\left(e^{-\mu \cdot t} \cdot x_t\right)$. For the evaluation of the expectation value and the variation of the Ornstein-Uhlenbeck process, we use the properties of the expectation value and the variance of Brownian process.

3 Conclusions

In this paper we used the technique of the integrating factor in the solving a two differential equations and are evaluated the expectation value and the variation for the solutions of these equations. The technique of integrating factor consist in finding an appropriate factor which multiplied by the both terms of stochastic differential equation and by arranging in such a way so that we obtain the stochastic differential. The Itô stochastic differential rules were used.

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Manuscript received: 02.06.2009 / accepted: 14.09.2009