ABOUT SOME PARTICULAR CLASSES OF BOUNDED OPERATORS ON PSEUDO-HILBERT SPACES II

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Abstract: *In this paper we study the gramian hyponormality and p-hyponormality for the operator* $\tilde{T} = \mid T \mid^{\frac{1}{2}} U \mid T \mid^{\frac{1}{2}}$ *when* T *is a particular gramian p-hyponormal operator on pseudo-Hilbert space and* $0 < p < 1$*. A property of the eigenspaces of* U *from the polar decomposition of an invertible gramian p-hyponormal operator* T *is established. Also other properties of the operators* U *and* $\mid T \mid$ *used in the polar decomposition of an invertible and gramian hyponormal operator* T *are presented. M. Ito proved a result concerning the power of p-hyponormal operators in [12], Theorem 1', which it is stated in pseudo-Hilbert spaces in Theorem 7.*

Keywords: pseudo-Hilbert space, classes of gramian hyponormal operators, Loynes space

1. Introduction

We recall that the notions of VE-spaces and VH-spaces (or LVH-spaces) respectively, were introduced by R. M. Loynes in [13] as generalizations of prehilbertian and Hilbert spaces, respectively. LVH-spaces have also been named pseudo-Hilbert spaces in [17] and later Loynes spaces in [4] and [18].

The pseudo-Hilbert spaces (Loynes spaces) are defined by the property of possessing an "inner product" (gramian) which takes its values in a suitable ordered topological space, instead of a scalar valued inner product, see [12], [13]. An important difference in these spaces is given by the falsity of the Riesz representation theorem which leads us to the absence of the existence of the gramian adjoint and of the gramian projections. Also the classical Cauchy-Schwarz inequality is not satisfied because the product $[x, x] \cdot [y, y], \; x, y \in Z$ does not exist.

The aim of this paper is to define and study some properties of the classes of gramian hyponormal, gramian p-hyponormal operators and gramian quasi-isometries, defined on pseudo-Hilbert spaces (Loynes spaces).

The structure of $\mathcal{B}^*(\mathcal{H})$ as a particular C^*-algebra (Corollary 1) allows us to recover many properties from Hilbert spaces in the case of these classes of operators. These properties are not true in every C^*-algebra. Some particular classes of gramian hyponormal operators resulting from the polar decomposition of an invertible operator or from the polar decomposition of an operator with U gramian unitary (when this decomposition exists) are given in Proposition 4 and Proposition 5.

As in Hilbert case by A. Aluthge (see [1]), the operator $\tilde{T} = \mid T \mid^{\frac{1}{2}} U \mid T \mid^{\frac{1}{2}}$ is gramian hyponormal with $\frac{1}{2} \leq p < 1$, see Theorem 2. For $0 < p < \frac{1}{2}$, the same characterization as in [1] is verified in Theorem 4. Theorem 3 gives another gramian hyponormal operators similarly to $\tilde{T}$ and leads us to a generalization of the Aluthge transform, in Theorem 8. In

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Theorem 5, the eigenspaces of a gramian unitary operator U from the polar decomposition of T reduce T as in Hilbert case, see [1].

Theorem 6, Corollary 1 and Corollary 2 were studied in Hilbert case by A. Aluthge and Wang in [2]. Theorem 7 and Corollary 4 were studied by M. Ito in the same case, see [11]. These properties studied by M. Ito generalize results on gramian p-hyponormal operators.

Definition 1([6]) *A locally convex space Z is called admissible in the Loynes sense if the following conditions are satisfied:*

(A.1) *Z is complete;*

(A.2) *there is a closed convex cone in Z, denoted Z_+, that defines an order relation on Z (that is $z_1 \leq z_2$ if $z_2 - z_1 \in Z_+$);*

(A.3) *there is an involution in Z, $Z \ni z \to z^* \in Z$ (that is $z^{**} = z$, $(\alpha z)^* = \overline{\alpha} z^*$, $(z_1 + z_2)^* = z_1^* + z_2^*$), such that $z \in Z_+$ implies $z^* = z$;*

(A.4) *the topology of Z is compatible with the order (that is there exists a basis of convex solid neighbourhoods of the origin);*

(A.5) *any monotonously decreasing sequence in Z_+ is convergent.*

Observation 1([6]) *A set $C \in Z$ is called solid if $0 \leq z' \leq z''$ and $z'' \in C$ implies $z' \in C$.*

Example 1([6])*$Z = C$, a C^*–algebra with topology and natural involution.*

Definition 2([6]) *Let Z be an admissible space in the Loynes sense. A linear topological space $\mathcal{H}$ is called pre-Loynes Z–space if it satisfies the following properties:*

(L1) *$\mathcal{H}$ is endowed with an Z–valued inner product (gramian), i.e. there exists an application $\mathcal{H} \times \mathcal{H} \ni (h, k) \to [h, k] \in Z$ having the properties:*

(G$_1$) *$[h, h] \geq 0$; $[h, h] = 0$ implies $h = 0$;*

(G$_2$) *$[h_1 + h_2, h] = [h_1, h] + [h_2, h]$;*

(G$_3$) *$[\lambda h, k] = \lambda [h, k]$;*

(G$_4$) *$[h, k]^* = [k, h]$;*

for all $h, k, h_1, h_2 \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.

(L2) *The topology of $\mathcal{H}$ is the weakest locally convex topology on $\mathcal{H}$ for which the application $\mathcal{H} \ni h \to [h, h] \in Z$ is continuous.*

Moreover, if $\mathcal{H}$ is a complete space with this topology, then $\mathcal{H}$ is called Loynes Z–space.

Example 2([6]) *Considering $Z = C$ in Example 1, Z with $[z_1, z_2] = z_2^* z_1$ is a Loynes-Z space.*

An important result which can be used below is given in the next statement, and was proved in ([6]).

Let $\mathcal{H}$ and $\mathcal{K}$ be two Loynes Z-spaces.

We recall that (see [12], [13], [6]) an operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called gramian bounded, if there exists a constant $\mu > 0$ such that in the sense of order of Z holds

(1.3.1) $$[Th, Th]_{\mathcal{K}} \leq \mu [h, h]_{\mathcal{H}}, \quad h \in \mathcal{H}.$$

We denote the class of such operators by $\mathcal{B}(\mathcal{H},\mathcal{K})$, and $\mathcal{B}^*(\mathcal{H},\mathcal{K}) = \mathcal{B}(\mathcal{H},\mathcal{K}) \cap \mathcal{L}^*(\mathcal{H},\mathcal{K})$. We also denote the introduced norm by

(1.3.2)
$$\|T\| = \inf\left\{\sqrt{\mu},\ \mu > 0 \text{ and satisfies (1.3.1)}\right\}.$$

Corollary 1([6]) *The space $\mathcal{B}^*(\mathcal{H},\mathcal{K})$ is a Banach space, and its involution $\mathcal{B}^*(\mathcal{H},\mathcal{K})$ in $\mathcal{B}^*(\mathcal{K},\ \mathcal{H})$ satisfies*
$$\|T^*T\| = \|T\|^2, \quad T \in \mathcal{B}^*(\mathcal{H},\mathcal{K}).$$
In particular $\mathcal{B}^*(\mathcal{H})$ is a C^*–algebra.

Lemma 1 ([6]) *If the operator $T \in \mathcal{B}^*(\mathcal{H})$ is invertible and there exists $T^{-1} \in \mathcal{B}^*(\mathcal{H})$ then T has a unique polar decomposition $T = UP$, where $P \in \mathcal{B}^*_+(\mathcal{H})$, and U is a gramian unitary operator from $\mathcal{B}^*(\mathcal{H})$.*

Theorem 1 *If $A \geq B \geq 0,\ A,\ B$ in $\mathcal{B}^*(\mathcal{H})$, then for each $r \geq 0$*

(i)
$$(B^r A^p B^r)^{\frac{1}{q}} \geq B^{\frac{(p+2r)}{q}}$$

and

(ii)
$$A^{\frac{p+2r}{q}} \geq (A^r B^p A^r)^{\frac{1}{q}}$$

hold for each p and q such that $p \geq 0,\ q \geq 1$ and $(1+2r)q \geq p + 2r$.

2. The main section

It is useful to recall the definition of gramian hyponormal operators and of gramian p-hyponormal operators.

Definition 3([8]) *An operator $T \in \mathcal{B}^*(\mathcal{H})$ is called gramian hyponormal if $T^*T \geq TT^*$.*

Definition 4 ([8])*An operator $T \in \mathcal{B}^*(\mathcal{H})$ is said to be gramian p-hyponormal if $(T^*T)^p \geq (TT^*)^p$, for $0 < p$.*

If $p = 1$, then T is gramian hyponormal.

Observation 2 ([8]) *Every gramian hyponormal operator is gramian p-hyponormal, $p \in (0, 1]$.*

Proposition 4 *Let $T = U \mid T \mid$ be an invertible gramian hyponormal operator. Then $U^n \mid T \mid$ is gramian hyponormal for any $n \geq 1,\ n \in \mathbb{N}$. Moreover, $U^n \mid T \mid$ is invertible in $\mathcal{B}^*(\mathcal{H})$ and also gramian hyponormal.*

The point (iii) of the following proposition is a generalization on pseudo-Hilbert spaces of Lemma 1 from the paper [10] of M. Fuji, C. Himeji and A. Matsumoto.

Proposition 5 (i) *Let $T = U \mid T \mid$ be an invertible gramian hyponormal operator, with $T = U \mid T \mid$ the polar decomposition of T. Then $\mid T \mid \geq U \mid T \mid U^*$ and the operator $U \mid T \mid^{\frac{1}{2}}$ is also gramian hyponormal.*

(ii) *Let* $T \in \mathcal{B}^*(\mathcal{H})$ *be invertible and the polar decomposition of* T, $T = U \mid T \mid$. *Then* T *is gramian n-hyponormal if and only if* $S = U \mid T \mid^n$, $n \in \mathbb{N}$ *is gramian hyponormal.*

(iii) *If* $T = U \mid T \mid$ *is the polar decomposition of* $T \in \mathcal{B}^*(\mathcal{H})$ *with* U *gramian unitary or* T *is invertible in* $\mathcal{B}^*(\mathcal{H})$ *and* $0 < p \leq 1$ *then* T *is gramian p-hyponormal if and only if* $S = U \mid T \mid^p$ *is gramian hyponormal.*

Proof: We shall prove only the last point. For the first two see [7].

(iii) "$\Longrightarrow$"

$$(T^*T)^p \geq (TT^*)^p \Leftrightarrow (\mid T \mid U^*U \mid T \mid)^p \geq (U \mid T \mid\mid T \mid U^*)^p \Leftrightarrow$$
$$\Leftrightarrow \mid T \mid^{2p} \geq (U \mid T \mid^2 U^*)^p = (U \mid T \mid U^*)^{2p} \Leftrightarrow$$
$$\Leftrightarrow S^*S = \mid T \mid^p U^*U \mid T \mid^p \geq \mid T^* \mid^{2p} =$$
$$= U \mid T \mid^{2p} U^* = U \mid T \mid^p \mid T \mid^p U^* = SS^*.$$

"$\Longleftarrow$"

$$S^*S = \mid T \mid^p U^*U \mid T \mid^p = \mid T \mid^{2p} = (\mid T \mid U^*U \mid T \mid)^p = (T^*T)^p \geq SS^* =$$
$$= U \mid T \mid^p \mid T \mid^p U^* = U \mid T \mid^{2p} U^* = \mid T^* \mid^{2p} = (U \mid T \mid^2 U^*)^p = (TT^*)^p.$$

□

The following result was given in Hilbert spaces by A. Aluthge in [1]. The existence of the polar decomposition with U gramian unitary is provided for the invertible operator in $\mathcal{B}^*(\mathcal{H})$, where $\mathcal{H}$ is a Loynes Z-space.

Theorem 2 *Let* $T = U \mid T \mid$ *be a gramian p-hyponormal operator* $\frac{1}{2} \leq p < 1$, *and* U *be gramian unitary. Then the operator* $\tilde{T} = \mid T \mid^{\frac{1}{2}} U \mid T \mid^{\frac{1}{2}}$ *is gramian hyponormal.*

A similar result with Theorem 2 can be stated below.

Theorem 3 *Let* $T = U \mid T \mid$ *be a gramian* $(\frac{p}{2})$*-hyponormal operator* $0 < p$, *and* U *be gramian unitary. Then the operator* $\tilde{T}_1 = \mid T \mid^{\frac{p}{2}} U \mid T \mid^{\frac{p}{2}}$ *is gramian hyponormal.*

Proof: By hypothesis $(T^*T)^{\frac{p}{2}} \geq (TT^*)^{\frac{p}{2}}$ that is $\mid T \mid^p \geq (U \mid T \mid^2 U^*)^{\frac{p}{2}} = = \mid T^* \mid^p = U \mid T \mid^p U^*$, where we used the equality, $\mid T^* \mid^2 = U \mid T \mid^2 U^*$.

Since U is gramian unitary and $\mid T \mid^p \geq 0$ we deduce

$$U^* \mid T \mid^p U \geq \mid T \mid^p \geq U \mid T \mid^p U^*.$$

This inequality implies,

$$\tilde{T}_1^*\tilde{T}_1 - \tilde{T}_1\tilde{T}_1^* = \mid T \mid^{\frac{p}{2}} U^* \mid T \mid^{\frac{p}{2}} \mid T \mid^{\frac{p}{2}} U \mid T \mid^{\frac{p}{2}} - \mid T \mid^{\frac{p}{2}} U \mid T \mid^{\frac{p}{2}} \mid T \mid^{\frac{p}{2}} U^* \mid T \mid^{\frac{p}{2}} =$$
$$= \mid T \mid^{\frac{p}{2}} U^* \mid T \mid^p U \mid T \mid^{\frac{p}{2}} - \mid T \mid^{\frac{p}{2}} U \mid T \mid^p U^* \mid T \mid^{\frac{p}{2}} =$$
$$= \mid T \mid^{\frac{p}{2}} (U^* \mid T \mid^p U - U \mid T \mid^p U^*) \mid T \mid^{\frac{p}{2}} \geq 0.$$

□

Theorem 4 *Let* $T = U \mid T \mid$ *be a gramian p-hyponormal operator,* $0 < p < \frac{1}{2}$, *and* U *be a gramian unitary. Then* $\tilde{T} = \mid T \mid^{\frac{1}{2}} U \mid T \mid^{\frac{1}{2}}$ *is gramian* $(p + \frac{1}{2})$*-hyponormal.*

Proof: Using the gramian p-hyponormality of T we have $(T^*T)^p \geq (TT^*)^p$ or by $U \mid T \mid^2 U^* = \mid T^* \mid^2$,

$$U^* \mid T \mid^{2p} U \geq \mid T \mid^{2p} \geq U \mid T \mid^{2p} U^*.$$

We consider $A = U^* \mid T \mid^{2p} U$, $B = \mid T \mid^{2p}$ and $C = U \mid T \mid^{2p} U^*$.

Then
$$(\tilde{T}^*\tilde{T})^{p+\frac{1}{2}} = (\mid T \mid^{\frac{1}{2}} U^* \mid T \mid U \mid T \mid^{\frac{1}{2}})^{p+\frac{1}{2}} = (B^{\frac{1}{4p}} A^{\frac{1}{2p}} B^{\frac{1}{4p}})^{p+\frac{1}{2}} \geq$$
$$\geq B^{(\frac{1}{2p}+\frac{2}{4p})(p+\frac{1}{2})} = B^{1+\frac{1}{2p}}$$
by Theorem 1, (i) using that $(1+\frac{2}{4p})\frac{2}{2p+1} = \frac{1}{p} = \frac{1}{2p} + \frac{2}{4p}$. Analogously,
$$(\tilde{T}\tilde{T}^*)^{p+\frac{1}{2}} = (\mid T \mid^{\frac{1}{2}} U \mid T \mid U^* \mid T \mid^{\frac{1}{2}})^{(p+\frac{1}{2})} = (B^{\frac{1}{4p}} C^{\frac{1}{2p}} B^{\frac{1}{4p}})^{p+\frac{1}{2}} \leq B^{(\frac{1}{2p}+\frac{2}{4p})(p+\frac{1}{2})} =$$
$= B^{1+\frac{1}{2p}}$ by (ii).

Therefore $(\tilde{T}^*\tilde{T})^{(p+\frac{1}{2})} \geq (\tilde{T}\tilde{T}^*)^{(p+\frac{1}{2})}$ that is $\tilde{T}$ is gramian $(p+\frac{1}{2})$-hyponormal.

□

Using the Theorem 2 and Theorem 4 we deduce,

Corollary 2 *If T is a gramian p-hyponormal operator with $0 < p < \frac{1}{2}$ and we suppose that there exists its polar decomposition, U being unitary, $T = U \mid T \mid$, then the operator $\mid \tilde{T} \mid^{\frac{1}{2}} \tilde{U} \mid \tilde{T} \mid^{\frac{1}{2}}$ is gramian hyponormal, where $\tilde{T} = \mid T \mid^{\frac{1}{2}} U \mid T \mid^{\frac{1}{2}}$ and $T = \tilde{U} \mid \tilde{T} \mid$ is polar decomposition of $\tilde{T}$.*

Theorem 5 *Let T be a gramian p-hyponormal operator, $0 < p < \frac{1}{2}$ and we suppose that $T = U \mid T \mid$ is its polar decomposition with U gramian unitary. Then the eigenspaces of U reduce T.*

Proof: It is sufficient to take only the case when $p = \frac{1}{2^n}$ for some n. Then $Q_n = (T^*T)^{\frac{1}{2^n}} - (TT^*)^{\frac{1}{2^n}} = \mid T \mid^{\frac{1}{2^{n-1}}} - U \mid T \mid^{\frac{1}{2^{n-1}}} U^* \geq 0$.

Be $M_\lambda = \{f \in \mathcal{H} : \ Uh = \lambda h, \mid \lambda \mid = 1\}$ the eigenspace of U which corresponds to eigenvalue λ. We obtain that $U^*h = \overline{\lambda}h$ for any $h \in M_\lambda$ by using the fact that U is gramian unitary. Thus we have,
$$[Q_n h, h] = [\mid T \mid^{\frac{1}{2^{n-1}}} h, h] - [U \mid T \mid^{\frac{1}{2^{n-1}}} U^*h, h] = [\mid T \mid^{\frac{1}{2^{n-1}}} h, h] - [\mid T \mid^{\frac{1}{2^{n-1}}} \overline{\lambda}h, U^*h] = 0,$$
and from $Q_n \geq 0$ we find $Q_n h = 0$, i.e. $U \mid T \mid^{\frac{1}{2^{n-1}}} U^*h = \mid T \mid^{\frac{1}{2^{n-1}}} h$ or $U \mid T \mid^{\frac{1}{2^{n-1}}} h = \lambda \mid T \mid^{\frac{1}{2^{n-1}}} h$. This implies $\mid T \mid^{\frac{1}{2^{n-1}}} h \in M_\lambda$, M_λ being thus an invariant space with respect to $\mid T \mid^{\frac{1}{2^{n-1}}}$ and therefore with respect to $\mid T \mid$. Thus the space M_λ reduces $\mid T \mid$ and then M_λ reduces T.

□

As a generalization in Loynes spaces of the Corollary 2.4, see [16], we find

Corollary 3 *If T is a gramian quasi-isometry which is quasinilpotent then $T = 0$.*

Proof: The spectral radius of $T \in \mathcal{B}^*(\mathcal{H})$, is by definition, $r(T) = \sup\{\mid \lambda \mid, \lambda \in \sigma(T)\} = 0$, using the definition of the quasinilpotent operator (as in Hilbert spaces, its spectrum $\sigma(T) = 0$).

But,
$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}},$$
see [17], so that $\|T^n\| \leq 1$ for some $n \in \mathbb{N}$ or $T^n = 0$.

If T is a gramian quasi-isometry, T^n will be also a gramian quasi-isometry and then $\|T^n\| \geq 1$ if $T^n \neq 0$. Thus $\|T^n\| = 1$. Using now, Proposition 3 from [7] we notice that T^n is gramian hyponormal. This fact implies that $\|T^n\| = r(T^n)$ according to Proposition 2, see [8] and then
$$1 = r(T^n) = \lim_{m\to\infty} \|(T^n)^m\|^{\frac{1}{m}} \leq \lim_{m\to\infty} \|T^m\|^{\frac{n}{m}} = r(T)^n = 0$$

so that $\|T^n\| = \|T\| = 0$ i.e. $T = 0$.

□

The well-known sequence of the operator inequalities, presented in [2], can be given also in our case, see

Theorem 6 *Let T be a gramian p-hyponormal invertible operator, $0 < p \leq 1$. The following inequalities take place for all positive integer n:*

$$(T^{n*}T^n)^{\frac{p}{n}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n*})^{\frac{p}{n}}.$$

Proof: If T is invertible operator then its polar decomposition $T = U \mid T \mid$ exists with U gramian unitary. For each $n \in \mathbb{N}$, we take $A_n = (T^{n*}T^n)^{\frac{p}{n}}$ and $B_n = (T^nT^{n*})^{\frac{p}{n}}$. By induction we shall establish the next inequalities

$$(1') \qquad A_n \geq A_1 \geq B_1 \geq B_n.$$

If $n = 1$ the inequalities (1') are obviously satisfied. We suppose that (1') takes place for n. Then, using that T is gramian p-hyponormal and $A_n \geq A_1$ we have

$$U^*A_nU \geq U^*A_1U \geq A_1,$$

as in the proof of Theorem 4, where $A_1 = (T^*T)^p = \mid T \mid^{2p}$.

Taking $C_n = (U^*A_n^{\frac{n}{p}}U)^{\frac{p}{n}}$ we find,

$$C_n = (U^*(T^{n*}T^n)^{\frac{p}{n}\cdot\frac{n}{p}}U)^{\frac{p}{n}} = (U^* \mid T^n \mid^2 U)^{\frac{p}{n}} = U^* \mid T^n \mid^{2\frac{p}{n}} U = U^*A_nU \geq A_1,$$

by using the Furuta's inequality. But, by induction, $B_n \leq B_1 \leq A_1$, then

$$B_{n+1} = (T^{n+1}T^{*n+1})^{\frac{p}{n+1}} = (T(T^nT^{*n})T^*)^{\frac{p}{n+1}} =$$

$$= (U \mid T \mid B_n^{\frac{n}{p}} \mid T \mid U^*)^{\frac{p}{n+1}} = U(\mid T \mid B_n^{\frac{n}{p}} \mid T \mid)^{\frac{p}{n+1}}U^* =$$

$$= U(A_1^{\frac{1}{2p}} B_n^{\frac{n}{p}} A_1^{\frac{1}{2p}})^{\frac{p}{n+1}}U^* \leq UA_1U^* = B_1,$$

by Furuta's inequality. Thus

$$A_{n+1} \geq A_1 \geq B_1 \geq B_{n+1}$$

which means by induction that (1') is true for $n \geq 1$.

□

Consequence 1 *If T is a gramian p-hyponormal operator with $0 < p \leq 1$, then T^n is gramian $(\frac{p}{n})$-hyponormal.*

The next result is a generalization of the Corollary 2, see [2] in Loynes spaces and it is also an extension of a result of Stampfli to gramian p-hyponormal operators.

Consequence 2 *If T is a gramian p-hyponormal operator with $0 < p \leq 1$ and T^n is gramian normal then T is gramian normal.*

Proof: By T^n is gramian normal, $T^nT^{n*} = T^{n*}T^n$.
Since

$$(T^{n*}T^n)^{\frac{p}{n}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n*})^{\frac{p}{n}}$$

we have $(T^*T)^p = (TT^*)^p$, i.e. $TT^* = T^*T$.

□

In 2001, M. Ito proved a more precise result concerning the power of p-hyponormal operators in [12], Theorem 1', extending a theorem of T. Furuta and M. Yanagida.

We can also state the same assertion in our case, using the Furuta's inequality from Theorem 1.

Theorem 7 *We consider T an invertible gramian p-hyponormal operator or a gramian p-hyponormal which admits polar decomposition $T = U \mid T \mid$ with U gramian unitary and $m \in \mathbb{N}$, $\mathrm{m}-1<\mathrm{p}\leq \mathrm{m}$. Then the following assertions hold:*

(1) $T^{n*}T^n \geq (T^nT^{n*})$ *and* $(TT^*)^n \geq T^nT^{n*}$ *take place for* $n = 1, 2, ..., m$.

(2) $(T^{n*}T^n)^{\frac{p+1}{n}} \geq (T^*T)^{p+1}$ *and* $(TT^*)^{p+1} \geq (T^nT^{n*})^{\frac{p+1}{n}}$ *take place for* $n = m+1, m+2, ...$

Proof: (1) We use the induction. We shall prove that

$$T^{n*}T^n \geq (T^nT^{n*}) \tag{3}$$

$$(TT^*)^n \geq T^nT^{n*} \tag{4}$$

for $n = 1, 2, ..., m$. The inequalities (3) and (4) are true for n=1. Assuming that (3) and (4) hold for some $n \leq m-1$, we have

$$T^{n*}T^n \geq (T^*T)^n \geq (TT^*)^n \geq T^nT^{n*}. \tag{5}$$

We also used that the gramian p-hyponormality, i.e. $(T^*T)^p \geq (TT^*)^p$, implies $(T^*T)^n = (T^*T)^{p\cdot\frac{n}{p}} \geq (TT^*)^{p\cdot\frac{n}{p}} = (TT^*)^n$, $\frac{n}{p} \in (0, 1]$.

From the last inequality, we obtain,

$$T^{n+1*}T^{n+1} = T^*(T^{n*}T^n)T \geq T^*(TT^*)^nT = (T^*T)^{n+1}$$

and also,

$$(TT^*)^{n+1} = T(T^*T)^nT^* \geq T(T^nT^{n*})T^* = T^{n+1}T^{n+1*}.$$

Therefore the check of (1) is satisfied, the inequalities (3) and (4) being true for $n+1 \leq m$.

(2) If $T = U \mid T \mid$ is the polar decomposition of T with U gramian unitary, then $\mid T^* \mid^2 = U \mid T \mid^2 U^*$ i.e. $\mid T^* \mid^2 \geq U \mid T \mid^2 U^*$ and $\mid T^* \mid^2 \leq U \mid T \mid^2 U^*$ with $(U \mid T \mid U^*)^2$, $\mid T \mid^2 \geq 0$.

Considering $f(x) = x^{\frac{1}{2}}$, $x \geq 0$ which is operator monotone, we deduce that $\mid T^* \mid \geq U \mid T \mid U^*$ and $\mid T^* \mid \leq U \mid T \mid U^*$ that is $\mid T^* \mid = U \mid T \mid U^*$ or $U^* \mid T^* \mid = \mid T \mid U^*$. But, $T^* = \mid T \mid U^*$ so that $T^* = U^* \mid T^* \mid$ is also the polar decomposition of T^* .

We shall check that

$$(T^{n*}T^n)^{\frac{p+1}{n}} \geq (T^*T)^{p+1} \quad \text{and} \tag{6}$$

$$(TT^*)^{p+1} \geq (T^nT^{n*})^{\frac{p+1}{n}} \tag{7}$$

for $n = m+1, m+2, ...$ We put $A = \mid T^n \mid^{\frac{2p}{n}}$ and $B = \mid T^{n*} \mid^{\frac{2p}{n}}$.

(i) First, we consider the case $n = m+1$. If $n = m$ then $\frac{p}{m} \leq 1$ and by the relations (1), (2) and the operator monotone function $f(x) = x^{\frac{p}{m}}$ we deduce,

$$(T^{m*}T^m)^{\frac{p}{m}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^mT^{m*})^{\frac{p}{m}} \tag{8}$$

using the p-hyponormality of T. This last inequality implies

$$A_m = (T^{m*}T^m)^{\frac{p}{m}} \geq (TT^*)^p = B_1 \text{ and } A_1 = (T^*T)^p \geq (T^mT^{m*})^{\frac{p}{m}} = B_m.$$

It is known that $(U^*DU)^r = U^*D^rU$ and $(UDU^*)^r = UD^rU^*$, $r > 0$, U being gramian unitary and $D \geq 0$.

Now, using Theorem 1, (i) for $\frac{m}{p} \geq 1$ and $\frac{1}{p} \geq 0$, we have,

$$(T^{m+1*}T^{m+1})^{\frac{p+1}{m+1}} = (U^* \mid T^* \mid T^{m*}T^m \mid T^* \mid U)^{\frac{p+1}{m+1}} =$$

$$= U^*(\mid T^* \mid T^{m*}T^m \mid T^* \mid)^{\frac{p+1}{m+1}} U = U^*(B_1^{\frac{1}{2p}} A_m^{\frac{m}{p}} B_1^{\frac{1}{2p}})^{\frac{1+\frac{1}{p}}{\frac{m}{p}+\frac{1}{p}}} U \geq$$

$$\geq U^* B_1^{1+\frac{1}{p}} U = U^* \mid T^* \mid^{2(p+1)} U = \mid T \mid^{2(p+1)} = (T^*T)^{p+1}.$$

i.e. exactly (6) for $n = m+1$.

The relation (7) for $n = m+1$ also holds by using Theorem 1, (ii) with $\frac{m}{p} \geq 1$ and $\frac{1}{p} \geq 0$, as below

$$(T^{m+1}T^{m+1*})^{\frac{p+1}{m+1}} = (U \mid T \mid T^mT^{m*} \mid T \mid U^*)^{\frac{p+1}{m+1}} =$$

$$= U(\mid T \mid T^mT^{m*} \mid T \mid)^{\frac{p+1}{m+1}} U^* = U(A_1^{\frac{1}{2p}} B_m^{\frac{m}{p}} A_1^{\frac{1}{2p}})^{\frac{1+\frac{1}{p}}{\frac{m}{p}+\frac{1}{p}}} U^* \leq$$

$$\leq U A_1^{1+\frac{1}{p}} U^* = U \mid T \mid^{2(p+1)} U^* = \mid T^* \mid^{2(p+1)} = (TT^*)^{p+1}.$$

(ii) Assuming that (6) and (7) are true for some $n \geq m+1$, we obtain for n,

$$(T^{n*}T^n)^{\frac{p}{n}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n*})^{\frac{p}{n}} \tag{9}$$

using again gramian p-hyponormality of T and that $f(x) = x^{\frac{p}{p+1}}$ is operator monotone. Analogously, we have

$$A_n = (T^{n*}T^n)^{\frac{p}{n}} \geq (TT^*)^p = B_1 \text{ and}$$
$$A_1 = (T^*T)^p \geq (T^nT^{n*})^{\frac{p}{n}} = B_n.$$

Using again Theorem 1, (i) for $\frac{n}{p} \geq 1$ and $\frac{1}{p} \geq 0$ we obtain that

$$(T^{n+1*}T^{n+1})^{\frac{p+1}{n+1}} \geq (T^*T)^{p+1}$$

if we consider instead of n, m in the proof of inequality (6) (i). So (6) holds for $n+1$.

Now, if we use Theorem 1, (ii) for $\frac{n}{p} \geq 1$ and $\frac{1}{p} \geq 0$ we obtain that

$$(T^{n+1}T^{n+1*})^{\frac{p+1}{n+1}} \leq (TT^*)^{p+1}$$

if we consider instead of n, m in the proof of inequality (7) (i). So (7) holds for $n+1$. By (i) and (ii), the proof (2) is complete.

□

Corollary 4 *Let $m \in \mathbb{N}$ and T be a gramian p-hyponormal operator for $m-1 < p \leq m$. Then the following statements are true:*

(a) *$T^{n*}T^n \geq T^nT^{n*}$ holds for all $n = 1, 2, ..., m-1$.*

(b) *$(T^{n*}T^n)^{\frac{p}{n}} \geq (T^nT^{n*})^{\frac{p}{n}}$ holds for all $n = m, m+1, ...$*

Proof: It easily results from the proof of Theorem 6.

□

The Aluthge transform, $\tilde{T}$, was generalized for any s and t such that $s \geq 0$ and $t \geq 0$. Thus $T(s,t) = \mid T \mid^s U \mid T \mid^t$. The following theorem shows the gramian hyponormality of $T(s,t)$. This is also a generalization of Theorem 3.

Theorem 8 *Let T be a gramian p-hyponormal operator, $p > 0$. If T is invertible or admits a polar decomposition, $T = U \mid T \mid$ with U gramian unitary, then for any s and t such that* $\max(s,t) \leq p$*, we have,*

$$T(s,t)^* T(s,t) \geq \mid T \mid^{2(s+t)} \geq T(s,t) T(s,t)^*$$

that is the Aluthge transform $T(s,t)$ of T is gramian hyponormal.

Proof: By using the hypothesis of gramian p-hyponormality, we have

$$\mid T \mid^{2p} \geq \mid T^* \mid^{2p} .$$

Taking $\max(s,t) \leq p$,

$$T(s,t)^* T(s,t) = \mid T \mid^t U^* \mid T \mid^{2s} U \mid T \mid^t = \mid T \mid^t U^* (\mid T \mid^{2p})^{\frac{s}{p}} U \mid T \mid^t \geq$$
$$\geq \mid T \mid^t U^* (\mid T^* \mid^{2p})^{\frac{s}{p}} U \mid T \mid^t = \mid T \mid^t U^* \mid T^* \mid^{2s} U \mid T \mid^t =$$
$$= \mid T \mid^t \mid T \mid^{2s} \mid T \mid^t = \mid T \mid^{2(s+t)} = \mid T \mid^s (\mid T \mid^{2p})^{\frac{t}{p}} \mid T \mid^s \geq$$
$$\geq \mid T \mid^s (\mid T^* \mid^{2p})^{\frac{t}{p}} \mid T \mid^s = \mid T \mid^s \mid T^* \mid^{2t} \mid T \mid^s =$$
$$= \mid T \mid^s U \mid T \mid^{2t} U^* \mid T \mid^s = T(s,t) T(s,t)^*.$$

We used again in two places that, $A \geq B \geq 0,\ A, B, X \in \mathcal{B}^*(\mathcal{H}) \Rightarrow X^* A X \geq X^* B X$ and $A^\alpha \geq B^\alpha,\ \alpha \in (0,1)$.

□

References

[1] Aluthge, A., *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory, **13** (1990), 307-315.

[2] Aluthge, A., Wang, D., *Powers of p-Hyponormal Operators*, J. of Inequal. Appl., 1999,Vol. **3**, pp. 297-284.

[3] Cassier, G., Suciu, L., *Mapping theorems and similarity to contractions, Hot Topics in Operator Theory*, pp.39-58,Theta 2008.

[4] Chobanyan, S.A., Weron, A., *Banach-space-valued stationary processes and their linear prediction, Dissertations Math.* , **125**, (1975), 1–45.

[5] Ciurdariu, L., Crăciunescu, A., *On Spectral Representation of Gramian Normal Operators on Pseudo-Hilbert Spaces*, Anal. Univ. de Vest Timişoara, Vol. XLV, (1), 2007, 131-149.

[6] Ciurdariu, L., *Clase de operatori liniari pe spaţii pseudo-hilbertiene şi aplicaţii, Teza de doctorat*, Universitatea de Vest, Timişoara, 2005.

[7] Ciurdariu, L., *About some particular classes of bounded operators on pseudo-Hilbert spaces*, preprint.

[8] Ciurdariu, L., *Hyponormal and p-hyponormal operators on pseudo-Hilbert spaces*, preprint.

[9] Conway, J. B., *Subnormal Operators*, Pitman, Boston, Mass.,(1981).

[10] Fujii, M., Himeji, C., Matsumoto, A., *Theorems of Ando and Saito for p-hyponormal operators*, Math. Japonica 39, No. **3** (1994), 595-598.

[11] Halmos, P. R., *A Hilbert Space Problem Book*, 2ndedn., Springer Verlag, New York, 1982.

[12] Ito, M., *Generalizations of the Results on Powers of p-Hyponormal Operators*, J. of Inequal. Appl.,2001,Vol. 6, pp. 1-15.
[13] Loynes, R.M., *Linear operators in VH-spaces*, Trans. American Math. Soc., **116**, (1965), 167-180.
[14] Loynes, R.M., *On generalized positive definite functions*, Proc. London Math. Soc., **3**, (1965), 373-384.
[15] Martin, M., Putinar, M., *Lectures on hyponormal operators*, Basel; Boston; Berlin; Birkhauser, 1989, Operator Theory, Vol. 39.
[16] Patel, S. M., *A note on quasi-isometries*, Glasnik Matematicki, **35** (2000), no. 55, 307-312.
[17] Strătilă, Ş., Zsido, L., *Operator Algebras*, Part I, II, T.U.T., Timişoara, 1995.
[18] Weron, A., Chobanyan, S.A., *Stochastic processes on pseudo-Hilbert spaces* (russian), Bull. Acad. Polon., Ser. Math. Astr. Phys., tom XX1, 9, (1973), 847-854.
[19] Weron, A., *Prediction theory in Banach spaces*, Proc. of Winter School on Probability, Karpacz, Springer Verlag, London, 1975, 207-228.

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