JOURNAL OF SCIENCE AND ARTS

COMBINATORICS PROBLEMS ON A TRIANGULAR BOARD

VASILE POP
Technical University of Cluj-Napoca, 25 George Baritiu Str., 400020, Cluj-Napoca, Romania,
e-mail: *vasile.pop@math.utcluj.ro*

Abstract: *In this paper, some combinatorics problems are presented, similar to those on a chess board, but on a triangular board.*

Keywords: triangular board, congruent equilateral triangles, border nodes

1 Introduction

The chess game is one of the oldest and well-known games, that imply special mental qualities. The fascination of the chessboard even caught others than chess players. The grid of squares, colored or not in black and white, constitutes a simple idea, based on which one can conceive many combinatorics problems that require a lot of ingenuity.

In the last years, similar problems on triangular boards started to appear (on these boards the squares became congruent equilateral triangles).

In this paper, we will try to present some problems and different types of reasoning on these boards, finite or infinite, most of them belonging to the author.

2 Definitions and notations

• We can obtain a finite triangular board in the following way:

- we divide each side of a triangle (the board) into n equal parts, the division points being called border nodes

-from each node on a side, we construct parallel lines to the other two sides(which end at other border nodes)

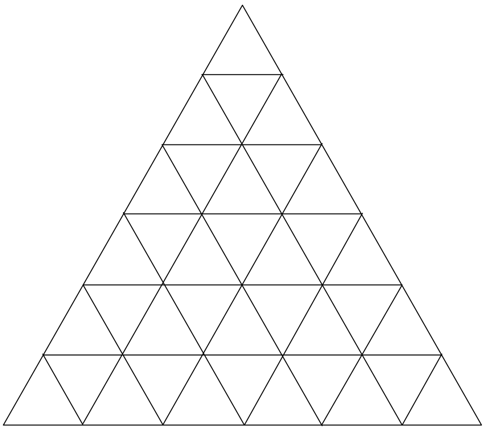
-the intersections of these $3(n-1)$ lines are called **nodes** and there are $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$ nodes

-the big triangle(the board) has been divided into $1+3+5+\cdots+(2n-1)=n^2$ smaller triangles, which we will call **fields**

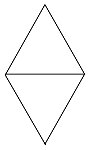
-two fields with a common side, will be called **neighbouring fields**

-two fields with a common vertex and the opposite sides in the two triangles parallel will be called **vertex fields**.

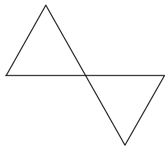
Paper presented at The VII-th International Conference on Nolinear Analysis and Applied Mathematics (ICNAAM), Târgovişte, 26-27 june, 2009



Triangular board

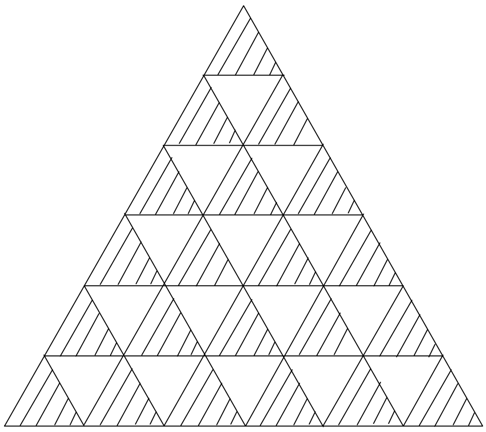


Neighbour fields



Vertex fields
(bow tie)

• A triangular "chessboard" is a board where the fields are colored into black and white, such that any two neighbouring fields have different colors.



Triangular chessboard, with 36 fields

• An **infinite triangular board** is a plane grid of congruent equilateral triangles(it is known that the only regular and congruent polygons that can pave the plane are the triangles, the squares and the hexagons).

- the lines on which the triangles of the grid are found, are called **grid lines**
- the small triangles are called **fields**
- the vertices of the triangles(intersections of the grid lines) are called **nodes**.

3 Problems on triangular boards

P.3.1. On every field of a triangular board with 100 fields, there is a snail. In every 10 seconds, each snail moves to a neighbouring field. Show that after 10 seconds, there will be at least 10 empty fields on the board.

P.3.2. For paving a room, shaped as an right isosceles triangle, with the catheters of length n (natural number), we use square plates, of side 1. They can be used as a whole, or split into right isosceles triangles with catheters of length 1. Which is the minimum number of splits necessary for paving the room?

P.3.3. A triangular board is made up of n^2 equilateral triangles of side length 1. Which is the minimum number of sides of length 1 that must be deleted so that no small triangle remains complete (with all its sides)?

P.3.4. On each field of a triangular board with n^2 fields are written numbers form the set $\{1, 2, \ldots, m\}$, $m, n \in \mathbb{N}^*$, such that any two consecutive numbers will be written in neighbouring fields. Show that $m \leq n^2 - n + 1$.

P.3.5. Find the minimum number of fields from an n^2 - fields board, that must be shaded, so that any other field will have a shaded neighbour.

P.3.6. The 64 fields of a triangular board are numbered from 1 to 64 (each number in a field). Show that for any numbering, there exist two neighbouring fields with the numbers difference at least 5.

P.3.7. We consider the nodes of a triangular board, made up of 64 fields. A "rook" placed in a node can capture all the others nodes situated on the lines parallel with the borders, that pass through that node. Which is the maximum number of "rooks" that can be placed in some of the nodes, such that no two "rooks" can capture each other?

P.3.8. In a plane grid of equilateral congruent triangles, the "knight" C_1 jumps from the acute vertex of a parallelogram of sides $l_1 = l_2 = 1$ to the opposite vertex, and the "knight" C_2 jumps from the acute vertex of a parallelogram of sides $l_1 = 1$, $l_2 = 2$ to the opposite vertex.

Determine all the nodes accessible to each "knight", starting from a fixed node.

P.3.9. Consider the plane grid of equilateral triangles of side length 1. A "pawn" situated inside a triangle can move into another triangle that shares a common edge with the first, and the opposite sides are parallel. We call "path", a finite succession of moves. Show that:

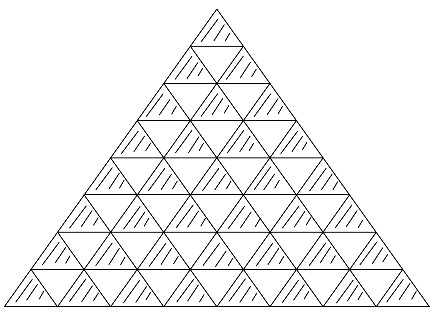
a) There is no path between two triangles that share a common side.

b) Out of any 5 triangles, we can pick two with a path between them.

4 Solutions to problems

S.4.1. We color the fields of board in black and white, like a chessboard, such that any two neighbouring fields should have different colors. We notice that the three fields situated in the corners of the board have the same color. Let that color be black. Then, we have $1 + 2 + \cdots + 10 = 55$ black triangles and $0 + 1 + 2 + \cdots + 9 = 45$ white triangles. Each 10 seconds, a snail switches the color, so after 10 seconds, in the black fields are only snails that came from white fields. Since we have only 45 "white snails" and 55 "black snails", we conclude that at least 10 black fields remain unoccupied.

S.4.2. We will show that the minimum number of split plates is $N = \left[\frac{n+1}{2}\right]$. We split each side in n equal parts and we partition the room into $1 + 3 + 5 + \cdots + (2n-1) = n^2$ right isosceles triangles with catheters of length 1, by constructing parallel lines to the sides of the triangle.



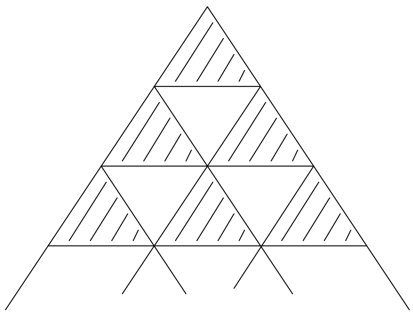
We color the triangle like the chessboard(the fields oriented towards the top with black, and the others with white). A whole plate, in any position, covers a white triangle and a black triangle. We notice that on the board, we have n more black triangles than white ones(one more on each line). It follows that for paving we need at least n black triangles which we can get by splitting $N = \left[\frac{n+1}{2}\right]$ plates.

We will now prove that if we have n triangular plates and enough square ones, we can pave the room(in two ways).

1. We place the n triangles on the hypotenuse of the room. There remain $n-1$ rectangles, of dimensions $1 \times (n-1)$, $1 \times (n-2), \ldots, 1 \times 1$ which can be paved with square plates.

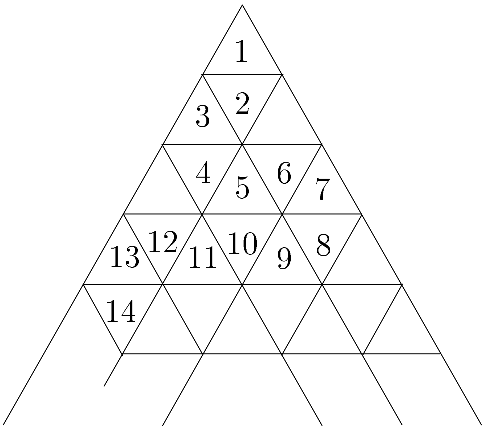
2. We can imagine that our room has been obtained by dividing a squared room of side length n, through a diagonal. If the big room is paved with n^2 squared plates, the diagonal splits exactly n of them, at the base of our room. Then, it is necessary to have only n triangles which can be obtained from $N = \left[\frac{n+1}{2}\right]$ squared plates.

S.4.3. We can color the small triangles in black and white, alternatively, like on a chess board.



By deleting a small side situated inside the big triangle, we suppress two triangles, a white one and a black one. Since we have n more black triangles than white ones, we need at least n small sides to be removed from the sides of the big triangle, removals that suppress a single triangle. This number is minimum and can be achieved by deleting a side of the big triangle and all the small sides parallel to it. The final number of sides that must be deleted is $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ small sides.

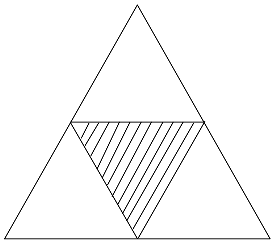
S.4.4. We color the fields in black and white (triangular "chessboard"), so that every neighbouring fields have different colors. If a field from a corner of the board is black, then all the fields from the corners are black and their number is $1+2+\cdots+n = \frac{n(n+1)}{2}$. The number of the white fields is $0+1+2+\cdots+(n-1) = \frac{n(n-1)}{2}$. In the given conditions, any two consecutive numbers are in different color fields, then in the sequence $1, 2, \ldots, m$, the numbers of terms from each color is the same, or differs by 1. Since the maximum number of white fields is $\frac{n(n-1)}{2}$, it follows that $m \leq 2\frac{n(n-1)}{2} + 1 = n^2 - n + 1$. For $m = n^2 - n + 1$ we can find a numbering: On the field from the top corner, we put 1, then we put 2, under it, on the second line. On its left field we put 3, and descending on line 3, we put 4 under 3. Then we move right, where we put 5, 6, 7, descend on line 4, where under 7 we put 8, and then we go left, putting 9, 10, 11, 12, 13. Finally, we descend on line 5, putting 14 under 13, we move right...



On each line, it remains exactly one unnumbered triangle, except for the first one, so we have $n-1$ triangles that do not contain numbers. We have numbered $n^2-(n-1)=n^2-n+1$ fields, with the numbers from 1 to n^2-n+1.

S.4.5. We will show that the minimum number of shaded fields is $\frac{n^2}{4}$ if n is even and $\frac{n^2+3}{4}$ if n is odd.

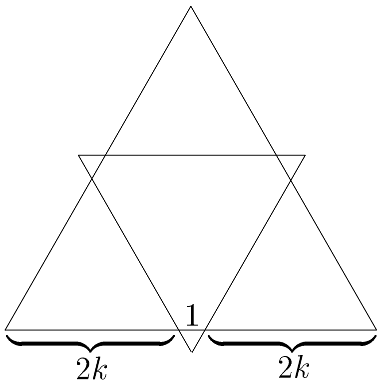
For n even, $n=2k$ we can partition the board into k^2 triangles, of 4 fields each. If we shade the central field of each triangle, we meet the condition.



Obviously, a shaded triangle cannot have more than three neighbours, so the minimum ratio between the number of shaded fields and the total number of fields is $\frac{1}{4}$, which means that k^2 is the minimum number of fields that must be shaded.

For n odd, we will treat separately the situations $n=4k+1$ and $n=4k+3$.

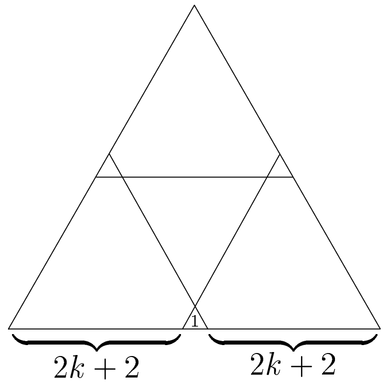
If $n=4k+1$, we make the partition from the figure:



in which we cut from the corners three triangles of side length $2k$. For each triangle from the corners, we need k^2 shaded fields. The total number is,

$$3k^2+(k+1)^2=4k^2+2k+1=\frac{16k^2+8k+4}{4}=\frac{n^2+3}{4}.$$

If $n=4k+3$, we cut from the corners, triangles of side length $2k+2$.



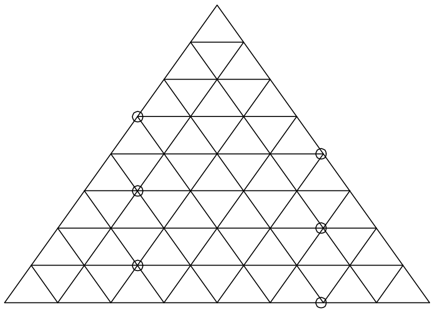
We need

$$(k+1)^2 + (k+1)^2 + (k+1)^2 + k^2 = 4k^2 + 6k + 3 = \frac{16k^2 + 24k + 12}{4} = \frac{n^2+3}{4}$$

shaded triangles.

S.4.6. We will call a path of length n, a sequence of triangles T_1, $T_2, \dots, T_n$ in which any two consecutive triangles are neighbours. Let's notice that any two triangles can be linked through a path of maximum length 15(15 is the minimum length of a path between the small triangles situated in the corners of the big triangle). We denote by T the triangle that contains the number 1, and by T' the triangle that contains the number 64, and we pick the shortest path from T to T': $T, T_2, T_3, \dots, T_k, T'$ where $k \leq 14$. If the difference between any two numbers situated into consecutive triangles is at most 4, then the number from T' would be at most $1 + 4k \leq 1 + 4 \cdot 14 = 57 < 64$, which is contradictory. It means that on the chosen path, there are at least two consecutive triangles(neighbours), for which the difference of the numbers is at least 5.

S.4.7.



We will show that the maximum number of rooks is 6. A placement with 6 rooks is presented in the figure. We will show that if we place 7 rooks, at least two of them can capture each other. We denote by $V_1, V_2, \dots, V_7$ the 7 nodes we pick, and by a_i, b_i, c_i, the distances from V_i to the sides of the big triangle BC, CA, AB, $i = \overline{1,7}$. It follows that $a_i + b_i + c_i = 8h$, where h is the height of a small triangle $\left(h = \frac{\sqrt{3}}{2}\right)$, and then

$$\sum_{i=1}^{7}(a_i + b_i + c_i) = 7 \cdot 8 \cdot h = 56 \cdot h.$$

On the other side

$$\sum_{i=1}^{7}(a_i + b_i + c_i) = \sum_{i=1}^{7} a_i + \sum_{i=1}^{7} b_i + \sum_{i=1}^{7} c_i.$$

If we presume that the 7 rooks cannot capture each other, it follows that the numbers $a_1, a_2, \ldots, a_7$ are all distinct, so

$$a_1 + a_2 + \cdots + a_7 \geq 0 + h + \cdots + 6h = 21h$$

and analogous

$$\sum_{i=1}^{7} b_i \geq 21h \text{ şi } \sum_{i=1}^{7} c_i \geq 21h.$$

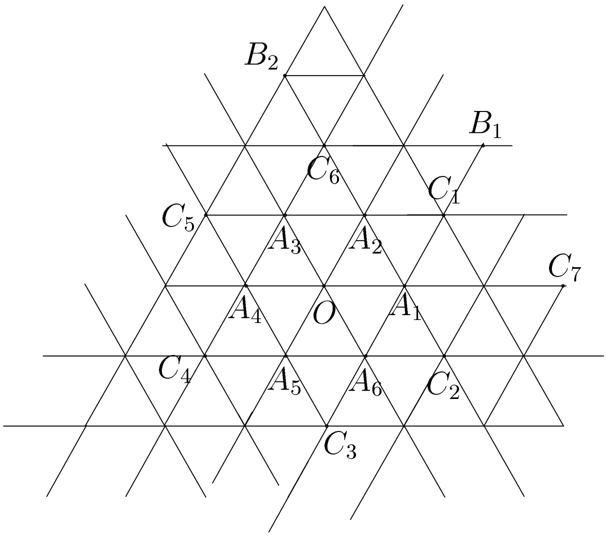
In conclusion

$$\sum_{i=1}^{7} a_i + \sum_{i=1}^{7} b_i + \sum_{i=1}^{7} c_i \geq 63h$$

and we obtain the contradiction $56h \geq 63h$, which shows that the hypothesis that the rooks cannot capture each other is false.

S.4.8. We pick a coordinate system in the grid plane, such that the set of the nodes to be

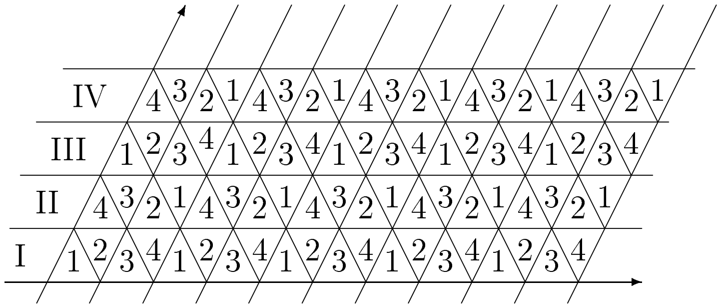
$$N = \left\{ \left(k, p\sqrt{3}\right), \left(k + \frac{1}{2}, p\sqrt{3} + \frac{\sqrt{3}}{2}\right) \mid k, p \in \mathbb{Z} \right\}.$$



a) From the origin, the knight C_1 can reach only the nodes with abscissa $x = \frac{3k}{2}$, $k \in \mathbb{Z}$, more precisely in the nodes of coordinates $\left(3k, p\sqrt{3}\right)$, $k, p \in \mathbb{Z}$ or $\left(\frac{1}{2} + (3k+1), p\sqrt{3} + \frac{\sqrt{3}}{2}\right)$, $k, p \in \mathbb{Z}$. Form O we can reach the points C_1 and C_5, situated on the vertical lines $x = \frac{3}{2}$ and $x = -\frac{3}{2}$. From O we can also reach with successive steps any point on the vertical line $x = 0$ (first on C_6 or C_3), and analogous, from any point, to any point on the same vertical line.

b)The knight C_2 can reach any node of the grid. First, we can move it in any neighbouring points: from O to B_1 and from B_1 to A_3, from O to B_2 and from B_2 to A_4. In the same way, we can reach from O to $A_1, A_2, A_3, A_4, A_5, A_6$. Obviously, in this way, continuing with the neighbouring points, we can reach any node of the grid.

S.4.9. We divide the grid into horizontal stripes of width $h = \frac{\sqrt{3}}{2}$, and we number the triangles as in the figure.



On the first stripe, we put 1, 2, 3, 4 which periodically repeats.

On the second stripe, we put 4, 3, 2, 1 which periodically repeats.

The third stripe repeats the first one, the fourth repeats the second, and so on (even stripes repeats the second stripe, and the odd ones repeats the first).

a) We notice that any move (from the first position) does not change the number inside the triangle. Since any two triangles with a common side have different numbers, it follows that we cannot reach one of them, starting from the other one.

b) Any 5 triangles, at least two of them are numbered the same. We will show that any two such triangles can be linked through a path.

Any two triangles, both numbered with 2 or 4 situated in the first two stripes can be linked by a path contained in these stripes. Also, from any triangle numbered with 2 or 4, we can reach the third stripe. The reasoning is the same for the triangles numbered with 1 or 3, situated in one of the stripes II or III. This way, we can bring a pawn from any stripe to one of the stripes I, II or III, then we can link them trough a path.

References

[1] Butter, M., *Baltic Wey*, Stockholm Univ., 1997.

[2] Barabanov E., Mazarik S., *Belarusian Mathematical Olympiad*, 2006.

[3] Iaglom A.M., Iaglom I.M., *Probleme neelementare tratate elementar*, Ed. Tehnică, 1981.

Manuscript received: 16.02.2009 / accepted: 24.08.2009