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ON THE HYERS-ULAM STABILITY OF THE ARITHMETIC AND GEOMETRIC MEANS

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Abstract: *We prove the Hyers-Ulam stability of the classical means using the stability of Jensen equation.*

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1 Introduction

One of the first results in the Hyers-Ulam stability theory of functional equation concerns Jensen equation, related to the stability of Cauchy equation.

We will use the result on stability of Jensen equation for proving the stability of some classic means (arithmetic and geometric mean).

Let $I \subset \mathbb{R}$ be an interval and $m : I \times I \to I$ a function with the property:

$$\min\{x, y\} \leq m(x, y) \leq \max\{x, y\} \text{ for all } x, y \in I.$$

Such a function is called mean on I.

The classical means are:

$$m_a(x, y) = \frac{x + y}{2} \quad \text{(arithmetic mean)}$$

$$m_g(x, y) = \sqrt{x \cdot y} \quad \text{(geometric means)}$$

$$m_h(x, y) = \frac{2xy}{x + y} \quad \text{(harmonic mean)}$$

$$m_2(x, y) = \sqrt{\frac{x^2 + y^2}{2}} \quad \text{(quadratic mean)}$$

Let $m : I \times I \to I$ be a mean.

Definition 1.1. We say that the function $F : I \to I$ is an invariant for the mean m if the following relation holds:

$$m(F(x), F(y)) = F(m(x, y)), \text{ for all } x, y \in I.$$

Let $\phi : I \times I \to [0, \infty)$ be a function with the property

$$\phi(x, x) = 0 \text{ for all } x \in I \text{ and } m : I \times I \to I \text{ a mean on } I.$$

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Definition 1.2. We say that the mean m is ϕ stable if for every $\varepsilon > 0$ and for every function $f : I \to I$ satisfying the inequality

$$\phi(f(m(x,y)), m(f(x), f(y))) \leq \varepsilon, \text{ for all } x, y \in I,$$

there exists a function $F : I \to I$ as invariant for the mean m and a number $\delta_\varepsilon > 0$ such that

$$\phi(F(x), f(x)) \leq \delta_\varepsilon \text{ for all } x \in I.$$

We recall a stability result for Jensen equation.

Definition 1.3. The functional equation

$$J : \begin{cases} f : \mathbb{R} \to \mathbb{R} \\ f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}, \; x, y \in \mathbb{R} \end{cases}$$

is called Jensen equation.

Theorem 1.4. [1], [2] *The function $f : \mathbb{R} \to \mathbb{R}$ satisfying Jensen equation are of the form*

$$f(x) = a(x) + b, \quad x \in \mathbb{R}$$

where $a : \mathbb{R} \to \mathbb{R}$ is an additive function ($a(x+y) = a(x) + a(y)$, $x, y \in \mathbb{R}$) and b is a real constant.

Theorem 1.5. [1], [2] *The continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying Jensen equation are*

$$f(x) = ax + b, \quad x \in \mathbb{R},$$

where $a, b \in \mathbb{R}$.

Definition 1.6. [4] We say that Jensen equation is stable in Hyers-Ulam sense if for every $\varepsilon > 0$ and every function $f : \mathbb{R} \to \mathbb{R}$ satisfying:

$$\left| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right| \leq \varepsilon, \text{ for all } x, y \in \mathbb{R}$$

there exists a number $\delta_\varepsilon > 0$ and a function $F : \mathbb{R} \to \mathbb{R}$ satisfying Jensen equation and the relation:

$$|F(x) - f(x)| \leq \delta_\varepsilon, \text{ for all } x \in \mathbb{R}.$$

Theorem 1.7. [3], [4] (Stability of Jensen equation) *For every $\varepsilon > 0$ and for every function $f : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\left| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right| \leq \varepsilon, \text{ for all } x, y \in \mathbb{R},$$

there exists a unique function $F : \mathbb{R} \to \mathbb{R}$ satisfying Jensen equation such that:

$$|F(x) - f(x)| \leq 2\varepsilon, \text{ for all } x \in \mathbb{R}.$$

Remark 1.8. • The function F from Theorem 1.7 is defined by

$$F(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} + f(0).$$

• The additive function $a : \mathbb{R} \to \mathbb{R}$ with the property

$$F(x) = a(x) + b, \quad x \in \mathbb{R}$$

is given by

$$a(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}.$$

• If in Theorem 1.7 the function f is continuous then

$$F(x)=ax+b,\quad x\in\mathbb{R},$$

where

$$a=\lim_{n\to\infty}\frac{f(2^n)}{2^n}\text{ and } b=f(0).$$

2 The stability of the arithmetic mean

We consider the arithmetic mean $m_a:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$,

$$m_a(x,y)=\frac{x+y}{2},\quad x,y\in\mathbb{R}$$

and we remark that the invariant function F for the arithmetic mean satisfies the equation:

$$F\left(\frac{x+y}{2}\right)=\frac{F(x)+F(y)}{2},\quad x,y\in\mathbb{R}.$$

We take the function $\phi_a:\mathbb{R}\times\mathbb{R}\to[0,\infty)$

$$\phi_a(x,y)=|x-y|$$

and we will show that the mean m_a is ϕ_a-stable in the sense of Definition 2.1.

The following results are consequences of Theorem 1.6 and Remark 1.7.

Theorem 2.1. *For every $\varepsilon>0$ and for every function $f:\mathbb{R}\to\mathbb{R}$ satisfying:*

$$|f(m_a(x,y))-m_a(f(x),f(y))|\le\varepsilon \text{ for all } x,y\in\mathbb{R},$$

there exists a unique function $F:\mathbb{R}\to\mathbb{R}$ satisfying Jensen equation

$$F(m_a(x,y))=m_a(F(x),F(y)),\ \text{for all } x,y\in\mathbb{R}$$

and

$$|F(x)-f(x)|\le 2\varepsilon,\ \text{for all } x\in\mathbb{R}.$$

Theorem 2.2. *For every $\varepsilon>0$ and for every continuous function $f:\mathbb{R}\to\mathbb{R}$ satisfying:*

$$|f(m_a(x,y))-m_a(f(x),f(y))|\le\varepsilon,\ \text{for all } x,y\in\mathbb{R}$$

there exists a unique pair of constants $(a,b)\in\mathbb{R}\times\mathbb{R}$ such that

$$|f(x)-ax-b|\le 2\varepsilon,\ \text{for all } x\in\mathbb{R}.$$

Remark 2.3. The constants a and b are defined by the relation

$$b=f(0)\text{ and } a=\lim_{n\to\infty}\frac{f(2^n)}{2^n}.$$

3 Stability of geometric mean

We consider the geometric mean

$$m_g : (0,\infty)\times(0,\infty)\to(0,\infty),\quad m_g(x,y)=\sqrt{xy},\quad x,y\in(0,\infty)$$

and define the function

$$\phi_g : (0,\infty)\times(0,\infty)\to[0,\infty),\quad \phi_g(x,y)=|\log x-\log y|,\quad x,y\in(0,\infty).$$

We will show that the geometric mean is ϕ_g-stable in the sense of Definition 2.1.

Remark 3.1. The function $F:(0,\infty)\to(0,\infty)$ is an invariant for the geometric mean iff

$$F(\sqrt{xy})=\sqrt{F(x)F(y)},\quad x,y\in(0,\infty).$$

If we define $H:\mathbb{R}\to\mathbb{R}$, $H=\log\circ F\circ\exp$, then H satisfies:

$$H\left(\frac{u+v}{2}\right)=\frac{H(u)+H(v)}{2},\quad u,v\in\mathbb{R}$$

so $H(u)=a(u)+b$, $u\in\mathbb{R}$, where $a:\mathbb{R}\to\mathbb{R}$ is an additive function and b is a real constant.

Thus F is of the form

$$F(x)=ce^{a(\log x)},\quad x\in(0,\infty),$$

where $c\in(0,\infty)$ is a constant and $a:\mathbb{R}\to\mathbb{R}$ is an additive function.

If F is continuous then $F(x)=cx^a$, $x\in(0,\infty)$, where $x\in(0,\infty)$ and $a\in\mathbb{R}$ are constants.

Theorem 3.1. *For every $\varepsilon>1$ and for every function $f:(0,\infty)\to(0,\infty)$ satisfying:*

$$\frac{1}{\varepsilon}\le\frac{f(m_g(x,y))}{m_g(f(x),f(y))}\le\varepsilon,\ \text{for all } x,y\in(0,\infty)\tag{1}$$

there exists a unique function $F:(0,\infty)\to(0,\infty)$ satisfying

$$F(m_g(x,y))=m_g(F(x),F(y)),\ \text{for all } x,y\in(0,\infty)$$

and

$$\frac{1}{\varepsilon^2}\le\frac{F(x)}{f(x)}\le\varepsilon^2,\ \text{for all } x\in(0,\infty).$$

Proof. The relation (1) is equivalent with

$$|\log f(m_g(x,y))-\log m_g(f(x),f(y))|\le\log\varepsilon$$

or

$$\phi_g(f(m_g(x,y)),m_g(f(x),f(y))|\le\log\varepsilon.$$

We consider the function $h:\mathbb{R}\to\mathbb{R}$, $h=\log\circ f\circ\exp$ and denoting $x=e^y$ and $y=e^v$ the inequality becomes:

$$\left|h\left(\frac{u+v}{2}\right)-\frac{h(u)+h(v)}{2}\right|\le\log\varepsilon,\ \text{for all } u,v\in\mathbb{R}.$$

So we are in the hypothesis of Theorem 1.6, then there exists a unique function $H:\mathbb{R}\to\mathbb{R}$ satisfying Jensen equation and more

$$|H(u)-h(u)|\le 2\log\varepsilon,\quad u\in\mathbb{R}.$$

We define the function $F:(0,\infty)\to(0,\infty)$, $F=\exp\circ H\circ\log$ and we have

$$|\log F(e^u)-\log f(e^u)|\le\log\varepsilon^2,\quad u\in\mathbb{R},$$

or

$$\frac{1}{\varepsilon^2}\le\frac{F(x)}{f(x)}\le\varepsilon^2,\ \text{for all } x=e^u\in(0,\infty)$$

(from Remark 3.1 the function F is an invariant for the geometric mean).

Corollary 3.2. *For every $\varepsilon>0$ and for every continuous function $f:(0,\infty)\to(0,\infty)$ with the property*

$$\frac{1}{\varepsilon}\le\frac{f(\sqrt{xy})}{\sqrt{f(x)f(y)}}\le\varepsilon,\ \textit{for all } x,y\in(0,\infty)$$

there exist a unique pair of real numbers (a,b) such that

$$\frac{1}{\varepsilon^2}\le b\cdot x^a\cdot f(x)\le\varepsilon^2,\ \textit{for all } x\in(0,\infty).$$

Proof. The continuous function $F:(0,\infty)\to(0,\infty)$ satisfying the equation

$$F(\sqrt{xy})=\sqrt{F(x)F(y)},\quad x,y\in(0,\infty)$$

are of the form

$$F(x)=a\cdot x^\beta,\quad x\in(0,\infty),$$

where $\alpha>0$ and $\beta\in\mathbb{R}$.

The relation

$$\frac{1}{\varepsilon^2}\le\frac{f(x)}{F(x)}\le\varepsilon^2,\quad x\in(0,\infty)$$

becomes

$$\frac{1}{\varepsilon^2}\le\alpha^{-1}\cdot x^{-\beta}\cdot f(x)\le\varepsilon^2,\quad x\in(0,\infty),$$

where $\alpha^{-1}=a$ and $-\beta=b$.

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