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SOME RESULTS ON GLOBAL SMOOTHNESS PRESERVATION BY STANCU-KANTOROVICH OPERATORS

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Abstract. *In this paper, we consider the Kantorovich-type modification of the operators introduced by D. D. Stancu in [11]*

$$S_{n,r,s}(f,x)=\sum_{j=0}^{n-rs}P_{n-rs,j}(x)\sum_{i=0}^{s}P_{s,i}(x)f\left(\frac{j+ir}{n}\right),$$

and we obtain some results relating to the preservation of global smoothness with respect to a certain second order modulus of continuity.

Keywords: *global smoothness preservation, second order modulus of continuity, Stancu operators.*

1. INTRODUCTION

Let $(L_n)_{n\in N}$, $N=\{1,2,\ldots\}$, a sequence of linear positive operators of approximation. If global smoothness of a function f is expressed by a Lipschitz condition with some modulus of continuity, it is of interest if $L_n f$ fulfill the same condition.

The Bernstein operators $B_n : C[0,1]\to C[0,1]$ are given by

$$B_n(f,x)=\sum_{j=0}^{n}P_{n,j}(x)f\left(\frac{j}{n}\right), \tag{1}$$

$f\in C[0,1], x\in[0,1]$, where $P_{n,j}(x)=\binom{n}{j}x^j(1-x)^{n-j}$, $C[0,1]$ being the Banach space of continuous real-valued function on $[0,1]$ with the Chebyshev norm. The preservation of global smoothness properties by these operators were studied in [6], [8], [4], [2], [5], [3]. It is well known that the Bernstein operators preserve the Lipschitz classes defined with the usual first order modulus of continuity

$$\omega_1(f,t)=\sup\{|f(y)-f(x)|: x,y\in[0,1], |y-x|\le t\}. \tag{2}$$

In [13], D.-X. Zhou showed that the Lipschitz classes with respect to the usual second order modulus

$$\omega_2(f,t)=\sup\{|f(x-h)-2f(x)+f(x+h)|: x\pm h\in[0,1], 0<h\le t\}, \tag{3}$$

are not preserved by the Bernstein operators. He introduced a new second order modulus of continuity

$$\tilde{\omega}_2(f,t)=\sup\{|f(x+h_1+h_2)-f(x+h_1)-f(x+h_2)+f(x)|: \tag{4}$$

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$$x, x+h_1+h_2 \in [0,1], h_1, h_2 > 0, h_1+h_2 \le 2t\}$$

and proved:

Theorem A. Let $f \in C[0,1]$, $n \in N$, $M > 0$ and $\alpha \in (0,1]$. If

$$\tilde{\omega}_2(f,t) \le Mt^{\alpha}, 0 < t \le \frac{1}{2},$$

then

$$\tilde{\omega}_2(B_n f,t) \le Mt^{\alpha}, 0 < t \le \frac{1}{2}.$$

Another global smoothness preservation result of this type can be obtained for the modulus ω_2^* introduced by R. PŁltŁnea [9], [10] and independently by J. Adell and J. de la Cal [1]

$$\omega_2^*(f,t) = \sup\{|(1-\lambda)f(x) + \lambda f(y) - f((1-\lambda)x + \lambda y)|: \\ 0 \le x < y \le 1, y - x \le 2t, \lambda \in [0,1]\}. \tag{5}$$

In [12], we established such result, in certain assumptions, for the Bernstein-type operators

$$L_n(f,x) = \sum_{j=0}^{n} P_{n,j}(x) F_{n,j}(f), f \in C[0,1], x \in [0,1], \tag{6}$$

where $F_{n,j} : C[0,1] \to R, j = \overline{0,n}$, are linear positive functionals. In particular, the result holds for the operators $K_n : L_1[0,1] \to C[0,1]$, $n \in N$, introduced by L. V. Kantorovich in [7]

$$K_n(f,x) = \sum_{j=0}^{n} P_{n,j}(x)(n+1) \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(u)du, \tag{7}$$

$f \in L_1[0,1], x \in [0,1]$, where $L_1[0,1]$ is the space of integrable functions.

In [11], D. D. Stancu introduced the generalized Bernstein operators $S_{n,r,s} : C[0,1] \to C[0,1]$, $n \in N$, $r,s \in N_0 = N \cup \{0\}$ such that $rs < n$, defined by

$$S_{n,r,s}(f,x) = \sum_{j=0}^{n-rs} P_{n-rs,j}(x) \sum_{i=0}^{s} P_{s,i}(x) f\left(\frac{j+ir}{n}\right), \tag{8}$$

$f \in C[0,1], x \in [0,1]$.

In this paper, we consider the Kantorovich-type modification of the Stancu operators and we study some preservation of global smoothness properties by these operators.

2. THE STANCU - KANTOROVICH OPERATORS

We consider the operators $K_{n,r,s} : L_1[0,1] \to C[0,1]$, $n \in N$, $r,s \in N_0$ (independent of $n \in N$) such that $rs < n$, defined for $f \in L_1[0,1]$ and $x \in [0,1]$ by

$$K_{n,r,s}(f,x) = \sum_{j=0}^{n-rs} P_{n-rs,j}(x) \sum_{i=0}^{s} P_{s,i}(x)(n+1) \int_{\frac{j+ir}{n+1}}^{\frac{j+ir+1}{n+1}} f(u)du. \tag{9}$$

We remark that

$$K_{n,r,s}(f,x)=\sum_{j=0}^{n-rs}P_{n-rs,j}(x)\sum_{i=0}^{s}P_{s,i}(x)\int_0^1 f\left(\frac{u+j+ir}{n+1}\right)du\,. \tag{10}$$

From definition, it is obvious that the operators $K_{n,r,s}$ are linear and positive. By a simple calculation, we obtain:

Lemma 1. For the test functions $e_k(x)=x^k$, $k\in\{0,1,2\}$, we have

$$K_{n,r,s}(e_0,x)=1; \tag{11}$$

$$K_{n,r,s}(e_1,x)=\frac{n}{n+1}x+\frac{1}{2(n+1)}; \tag{12}$$

$$K_{n,r,s}(e_2,x)=\frac{n^2-n+rs-r^2s}{(n+1)^2}x^2+\frac{2n-rs+r^2s}{(n+1)^2}x+\frac{1}{3(n+1)^2}\,. \tag{13}$$

Since for $r,s\in N_0$ fixed, $\lim\limits_{n\to\infty}K_{n,r,s}(e_k)=e_k$ in $C[0,1]$, $k\in\{0,1,2\}$, by well known Korovkinôs theorem, we obtain:

Theorem 1. If $r,s\in N_0$ are fixed, then $\lim\limits_{n\to\infty}K_{n,r,s}(f)=f$ uniformly on $[0,1]$, for every $f\in C[0,1]$.

Further we give a useful representation of Kantorovich-Stancu operators. Following the ideas from [13], [4], for the Bernstein-type operators (6) we obtain (see [12]) for $0\le x<y\le 1$ and $\lambda\in[0,1]$

$$L_n(f,(1-\lambda)x+\lambda y)=\sum_{k+l=0}^{n}P_{n,k,l}(x,y-x)\sum_{m=0}^{l}P_{l,m}(\lambda)F_{n,k+m}(f), \tag{14}$$

where $P_{n,k,l}(u,v)=\dfrac{n!}{k!l!(n-k-l)!}u^kv^l(1-u-v)^{n-k-l}$ are the two-variable Bernstein basis polynomials, $u,v\ge 0$, $u+v\le 1$.

By repeated application of this representation (suitable) we will obtain:

Lemma 2. Let $f\in L_1[0,1]$, $n\in N$, $r,s\in N_0$ (independent of $n\in N$) such that $rs<n$, $0\le x<y\le 1$ and $\lambda\in[0,1]$. Then

$$\begin{aligned}K_{n,r,s}(f,(1-\lambda)x+\lambda y)&=\sum_{k_1+l_1=0}^{s}\sum_{k_2+l_2=0}^{n-rs}P_{s,k_1,l_1}(x,y-x)P_{n-rs,k_2,l_2}(x,y-x)\cdot\\&\cdot\sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2}P_{l_1,m_1}(\lambda)P_{l_2,m_2}(\lambda)\int_0^1 f\left(\frac{u+k_2+m_2+(k_1+m_1)r}{n+1}\right)du\,.\end{aligned} \tag{15}$$

Proof:

$$\begin{aligned}K_{n,r,s}(f,(1-\lambda)x+\lambda y)&=\sum_{j=0}^{n-rs}P_{n-rs,j}((1-\lambda)x+\lambda y)\sum_{i=0}^{s}P_{s,i}((1-\lambda)x+\lambda y)\int_0^1 f\left(\frac{u+j+ir}{n+1}\right)du\\&=\sum_{j=0}^{n-rs}P_{n-rs,j}((1-\lambda)x+\lambda y)\sum_{k_1+l_1=0}^{s}P_{s,k_1,l_1}(x,y-x)\sum_{m_1=0}^{l_1}P_{l_1,m_1}(\lambda)\int_0^1 f\left(\frac{u+j+(k_1+m_1)r}{n+1}\right)du\\&=\sum_{k_1+l_1=0}^{s}P_{s,k_1,l_1}(x,y-x)\sum_{m_1=0}^{l_1}P_{l_1,m_1}(\lambda)\sum_{j=0}^{n-rs}P_{n-rs,j}((1-\lambda)x+\lambda y)\int_0^1 f\left(\frac{u+j+(k_1+m_1)r}{n+1}\right)du\end{aligned}$$

$$= \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} P_{s,k_1,l_1}(x, y-x) P_{n-rs,k_2,l_2}(x, y-x) \cdot$$
$$\cdot \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} P_{l_1,m_1}(\lambda) P_{l_2,m_2}(\lambda) \int_0^1 f\left(\frac{u+k_2+m_2+(k_1+m_1)r}{n+1}\right) du .$$

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3. GLOBAL SMOOTHNESS PRESERVATION

We now state a result concerning the preservation property of Lipschitz classes by the Stancu-Kantorovich operators.

Theorem 1. Let $f \in C[0,1]$, $n \in N$, $r,s \in N_0$ (independent of $n \in N$) such that $rs < n$, $M > 0$ and $\alpha \in (0,1]$. If

$$\omega_1(f,t) \le Mt^{\alpha}, 0 < t \le 1,$$

then

$$\omega_1(K_{n,r,s} f,t) \le Mt^{\alpha}, 0 < t \le 1.$$

Proof: Let $x, y \in [0,1]$ be such that $|x-y| \le t$. We can assume that $x < y$.
From (15) we have

$$K_{n,r,s}(f,y) = \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} P_{s,k_1,l_1}(x, y-x) P_{n-rs,k_2,l_2}(x, y-x) \cdot \int_0^1 f\left(\frac{u+k_2+l_2+(k_1+l_1)r}{n+1}\right) du$$

and

$$K_{n,r,s}(f,x) = \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} P_{s,k_1,l_1}(x, y-x) P_{n-rs,k_2,l_2}(x, y-x) \cdot \int_0^1 f\left(\frac{u+k_2+k_1 r}{n+1}\right) du .$$

Then

$$|K_{n,r,s}(f,y) - K_{n,r,s}(f,x)|$$
$$\le \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} P_{s,k_1,l_1}(x, y-x) P_{n-rs,k_2,l_2}(x, y-x) \cdot$$
$$\cdot \int_0^1 \left| f\left(\frac{u+k_2+l_2+(k_1+l_1)r}{n+1}\right) - f\left(\frac{u+k_2+k_1 r}{n+1}\right) \right| du$$
$$\le \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} P_{s,k_1,l_1}(x, y-x) P_{n-rs,k_2,l_2}(x, y-x) \cdot \omega_1\left(f, \frac{l_2+l_1 r}{n+1}\right)$$
$$\le \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} P_{s,k_1,l_1}(x, y-x) P_{n-rs,k_2,l_2}(x, y-x) M\left(\frac{l_2+l_1 r}{n+1}\right)^{\alpha}$$
$$\le M\left(\sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} P_{s,k_1,l_1}(x, y-x) P_{n-rs,k_2,l_2}(x, y-x) \frac{l_2+l_1 r}{n+1}\right)^{\alpha}$$
$$= M\left(\frac{n-rs}{n+1}(y-x) + \frac{rs}{n+1}(y-x)\right)^{\alpha} \le M\left(\frac{n}{n+1}\right)^{\alpha} t^{\alpha} < Mt^{\alpha} .$$

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We give now a result related to global smoothness preservation with respect to the second order modulus given in (5).

Theorem 2. Let $f \in C[0,1]$, $n \in N$, $r,s \in N_0$ (independent of $n \in N$) such that $rs < n$, $M > 0$ and $\alpha \in (0,1]$. If

$$\omega_2^*(f,t) \leq Mt^\alpha, 0 < t \leq \frac{1}{2},$$

then

$$\omega_2^*(K_{n,r,s} f,t) \leq Mt^\alpha, 0 < t \leq \frac{1}{2}.$$

Proof: Let $0 < t \leq \frac{1}{2}$, $x, y \in [0,1]$ such that $x < y, y - x \leq 2t$ and $\lambda \in [0,1]$. We have $\left|(1-\lambda)K_{n,r,s}(f,x) + \lambda K_{n,r,s}(f,y) - K_{n,r,s}(f,(1-\lambda)x + \lambda y)\right|$

$$\leq \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} P_{s,k_1,l_1}(x,y-x) P_{n-rs,k_2,l_2}(x,y-x) \cdot$$

$$\cdot \left| (1-\lambda) \int_0^1 f\left(\frac{u+k_2+k_1 r}{n+1}\right) du + \lambda \int_0^1 f\left(\frac{u+k_2+l_2+(k_1+l_1)r}{n+1}\right) du \right.$$

$$\left. - \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} P_{l_1,m_1}(\lambda) P_{l_2,m_2}(\lambda) \int_0^1 f\left(\frac{u+k_2+m_2+(k_1+m_1)r}{n+1}\right) du \right|$$

$$\leq \sum_{\substack{k_1+l_1=0 \\ l_2+rl_1 \neq 0}}^{s} \sum_{k_2+l_2=0}^{n-rs} P_{s,k_1,l_1}(x,y-x) P_{n-rs,k_2,l_2}(x,y-x) \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} P_{l_1,m_1}(\lambda) P_{l_2,m_2}(\lambda) \cdot$$

$$\cdot \int_0^1 \left| \left(1 - \frac{m_2+rm_1}{l_2+rl_1}\right) f\left(\frac{u+k_2+rk_1}{n+1}\right) + \frac{m_2+rm_1}{l_2+rl_1} f\left(\frac{u+k_2+l_2+(k_1+l_1)r}{n+1}\right) \right.$$

$$\left. - f\left(\frac{u+k_2+m_2+(k_1+m_1)r}{n+1}\right) \right| du$$

$$\leq \sum_{\substack{k_1+l_1=0 \\ l_2+rl_1 \neq 0}}^{s} \sum_{k_2+l_2=0}^{n-rs} P_{s,k_1,l_1}(x,y-x) P_{n-rs,k_2,l_2}(x,y-x) \cdot \omega_2^*\left(f, \frac{l_2+rl_1}{2(n+1)}\right)$$

$$\leq \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} P_{s,k_1,l_1}(x,y-x) P_{n-rs,k_2,l_2}(x,y-x) M\left(\frac{l_2+rl_1}{2(n+1)}\right)^\alpha$$

$$\leq M\left(\sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} P_{s,k_1,l_1}(x,y-x) P_{n-rs,k_2,l_2}(x,y-x) \frac{l_2+l_1 r}{2(n+1)}\right)^\alpha$$

$$= M\left(\frac{n-rs}{2(n+1)}(y-x) + \frac{rs}{2(n+1)}(y-x)\right)^\alpha \leq M\left(\frac{n}{n+1}\right)^\alpha t^\alpha < Mt^\alpha.$$

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