ORIGINAL PAPER

AN ALTERNATIVE PROOF OF A TWO-SIDED ESTIMATE OF e^x-$(1+x/t)^t$

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Abstract. *In this paper we give an alternative proof of a two sided estimate of* $e^x-(1+x/t)^t$*, stated in [1, Theorem 1]. With this proof, the conditions of this theorem are improved, as demonstrated by a different way in [3].*
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1. INTRODUCTION

The main result in [1] is the following theorem

Theorem 1.
i) If $x>0$, $t>0$ and $t>(1-x)/2$, then

$$\frac{x^2e^x}{2t+x+\max(x,x^2)}<e^x-\left(1+\frac{x}{t}\right)^t<\frac{x^2e^x}{2t+x}. \tag{1}$$

ii) If $x>0$, $t>0$ and $t>(x-1)/2$, then

$$\frac{x^2e^{-x}}{2t-x+x^2}<e^{-x}-\left(1-\frac{x}{t}\right)^t<\frac{x^2e^{-x}}{2t-2x+\min(x,x^2)}. \tag{2}$$

This theorem was established in [1] by studying the sign of some specific functions (related to inequalities (1)) and with the help of the harmonic, logarithmic and arithmetic mean inequality.

An attempt for another proof of Theorem 1 was given in [2]. In fact, the authors in [2] had the idea to exploit an elementary consequence of Lagrange Theorem to derive the double inequality

$$x\left(e-\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)\left(\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^{x-1}<e^x-\left(1+\frac{x}{t}\right)^t<x\left(e-\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)e^{x-1}, \tag{3}$$

and to use it together with the estimate

$$\frac{ex}{2t+2x}<e-\left(1+\frac{x}{t}\right)^{\frac{t}{x}}<\frac{ex}{2t+x} \qquad (x,t>0) \tag{4}$$

in order to give a simple proof of (1).

But as explained in [3], the estimate (3) is valid only for $x>1$ and that for $x<1$ it must be reversed. Accordingly, the right (resp. left) side inequality of (1) is a simple consequence of (3) and (4) only for $x>1$ (resp. $x<1$).

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In [3] we showed that for $t > 0$ and $x > \alpha > 0$ it holds

$$\frac{x}{\alpha}\left(e^{\alpha}-\left(1+\frac{x}{t}\right)^{\frac{\alpha t}{x}}\right)\left(\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^{x-\alpha} < e^{x}-\left(1+\frac{x}{t}\right)^{t} < \frac{x}{\alpha}\left(e^{\alpha}-\left(1+\frac{x}{t}\right)^{\frac{\alpha t}{x}}\right)e^{x-\alpha}. \qquad (5)$$

For $\alpha = 1$, estimate (3) is a particular case of (5). Also, when α approaches zero in (5), we find the following double inequality (valid for $t > 0$ and $x > 0$)

$$x\left(1+\frac{x}{t}\right)^{t}\left(1-\frac{t}{x}\ln\left(1+\frac{x}{t}\right)\right) < e^{x}-\left(1+\frac{x}{t}\right)^{t} < xe^{x}\left(1-\frac{t}{x}\ln\left(1+\frac{x}{t}\right)\right). \qquad (6)$$

The main application of (6) is the improvement of the conditions of Theorem 1 in case i), since the upper bound of (6) is better than the upper bound of (1) for $x, t > 0$. This motivates us to search for an alternative proof of Theorem 1 which takes into account the new extra conditions. This alternative proof is presented in the next section for this new formulation of Theorem 1 (see [3, Theorem 2])

Theorem 2. If $x \neq 0$ and $t > \max\ (0, -x)$, then

$$\frac{x^2e^x}{2t+x+\max\ (x,x^2)} < e^{x}-\left(1+\frac{x}{t}\right)^{t} < \frac{x^2e^x}{2t+2x+\min\ (-x,x^2)}. \qquad (7)$$

2. MAIN RESULT

In this section we give a proof of Theorem 2 different from that given in [1] for Theorem 1. Recall that Theorem 2 is equivalent to Theorem 1 with its conditions replaced by the extra conditions ($x \neq 0$ and $t > \max\ (0, \ x)$) and the estimates ((1), (2)) replaced by the estimate (7).

We begin by writing the following four estimates which are equivalent to estimate (7).

a) If $0 < x < 1$ and $t > 0$, then

$$\frac{x^2e^x}{2t+2x} < e^{x}-\left(1+\frac{x}{t}\right)^{t} < \frac{x^2e^x}{2t+x}. \qquad (8)$$

b) If $x > 1$ and $t > 0$, then

$$\frac{x^2e^x}{2t+x+x^2} < e^{x}-\left(1+\frac{x}{t}\right)^{t} < \frac{x^2e^x}{2t+x}. \qquad (9)$$

c) If $-1 < x < 0$ and $t > -x$, then

$$\frac{x^2e^x}{2t+x+x^2} < e^{x}-\left(1+\frac{x}{t}\right)^{t} < \frac{x^2e^x}{2t+2x+x^2}. \qquad (10)$$

d) If $x < -1$ and $t > -x$, then

$$\frac{x^2e^x}{2t+x+x^2} < e^{x}-\left(1+\frac{x}{t}\right)^{t} < \frac{x^2e^x}{2t+x}. \qquad (11)$$

Now observe that:

The right-hand side inequalities of (8), (9), and (11) are equivalent to

$$\frac{2t+x-x^2}{2t+x}e^{x} < \left(1+\frac{x}{t}\right)^{t}, \qquad (12)$$

the right-hand side inequality of (10) is equivalent to

$$\frac{2t+2x}{2t+2x+x^2}e^x < \left(1+\frac{x}{t}\right)^t, \tag{13}$$

the left-hand side inequalities of (9), (10) and (11) are equivalent to

$$\frac{2t+x}{2t+x+x^2}e^x > \left(1+\frac{x}{t}\right)^t, \tag{14}$$

and finally, the left-hand side inequality of (8) is equivalent to

$$\frac{2t+2x-x^2}{2t+2x}e^x > \left(1+\frac{x}{t}\right)^t. \tag{15}$$

Remark that (12) is valid if its lower bound is negative. So, henceforth we suppose this lower bound positive. Also, we can check easily that the lower bound of (13) and the upper bounds of (14) and (15) are positive under the corresponding conditions. Thus, we can apply the Logarithm function to (12), (13), (14) and (15) to get, respectively,

$$f_1(t,x) := t\ln\left(1+\frac{x}{t}\right) - \ln\left(\frac{2t+x-x^2}{2t+x}\right) - x > 0,$$

$$f_2(t,x) := t\ln\left(1+\frac{x}{t}\right) - \ln\left(\frac{2t+2x}{2t+2x+x^2}\right) - x > 0,$$

$$f_3(t,x) := -t\ln\left(1+\frac{x}{t}\right) + \ln\left(\frac{2t+x}{2t+x+x^2}\right) + x > 0$$

and

$$f_4(t,x) := -t\ln\left(1+\frac{x}{t}\right) + \ln\left(\frac{2t+2x-x^2}{2t+2x}\right) + x > 0.$$

To study the sign of the functions f_i, we consider x as the variable and t as a parameter. The derivatives of the functions f_i with respect to x, denoted f_i', are given by

$$f_1' = \frac{x^2(x^2+2tx+t)}{(t+x)(2t+x)(2t+x-x^2)},$$

$$f_2' = -\frac{x^3(x+1)}{(t+x)(2t+2x+x^2)},$$

$$f_3' = \frac{x^2(x^2+2tx-t)}{(t+x)(2t+x)(2t+x+x^2)},$$

and

$$f_4' = \frac{x^2(1-x)}{(t+x)(2t+2x-x^2)}.$$

Now we are in position to study the sign of the functions f_i.

1) For f_1 we have:

If $0 < x < 1$ and $t > 0$ or $x > 1$ and $t > 0$, then f_1' is positive. So, f_1 increases and as $f_1(t,0) = 0$ we get $f_1 > 0$.

If $x < -1$ and $t > -x$, then $-t < x < -1$ and $t > 1$. We see that the polynomial $x^2 + 2tx + t$ increases for $x > -t$. So, with $-t < x < -1$ and $t > 1$, we have

$$x^2 + 2tx + t < 1 - t < 0.$$

Thus, $f_1 > f_1(t,-1) = t\ln\left(1-\frac{1}{t}\right) - \ln\left(\frac{2t-2}{2t-1}\right) + 1$ since f_1 decreases when $-t < x < -1$. The second derivative of $f_1(t,-1)$ has the form

$$\frac{1}{t(2t-1)^2(t-1)}.$$

Evidently, $f_1(t,-1)$ is concave for $t > 1$, with $f_1(\infty,-1) = 0$, so $f_1 > f_1(t,-1) > 0$ for $t > 1$.

2) For f_2 we have:

If $-1 < x < 0$ and $t > -x$, then $f_2' < 0$. We conclude that f_2 is decreasing whenever $-t < x < 0$, hence $f_2 > f_2(t,0) = 0$.

3) For f_3 we have:

If $x > 1$ and $t > 0$, then $x^2 + 2tx - t > 1 + t > 0$ yields $f_3' > 0$. So, f_3 increases and we get $f_3 > f_3(t,1) = -t\ln\left(1+\frac{1}{t}\right) + \ln\left(\frac{2t+1}{2t+2}\right) + 1$. The function $f_3(t,1)$ is positive for $t > 0$ since $f_3(\infty,1) = 0$ and is concave with second derivative given by

$$\frac{1}{t(2t+1)^2(t+1)}.$$

If $-1 < x < 0$ and $t > -x$, then $-t < x < 0$ and $t > 0$. The polynomial $x^2 + 2tx - t$ increases for $x > -t$. So, for $-t < x < 0$ and with $t > 0$ we have

$$x^2 + 2tx - t < -t < 0.$$

This shows that f_3 decreases whenever $-t < x < 0$, which gives $f_3 > f_3(t,0) = 0$.

If $x < -1$ and $t > -x$, then $-t < x < -1$ and $t > 1$. We use again the monotony of the polynomial $x^2 + 2tx - t$ to get

$$x^2 + 2tx - t < 1 - 3t < -2.$$

So, f_3 decreases in the interval $-t < x < -1$, which yields

$$f_3 > f_3(t,-1) = -t\ln\left(1-\frac{1}{t}\right) + \ln\left(\frac{2t-1}{2t}\right) - 1.$$

The second derivative of $f_3(t,-1)$ has the form

$$\frac{5t(t-1)+1}{t^2(2t-1)^2(t-1)^2}.$$

We see that the function $f_3(t,-1)$ is concave, with $f_3(\infty,-1) = 0$, so $f_3 > f_3(t,-1) > 0$ for $t > 1$.

4) For f_4 we have:

If $0 < x < 1$ and $t > 0$, then $f_4' > 0$ and we conclude that f_4 is increasing in the interval $0 < x < 1$. So, $f_4 > f_4(t,0) = 0$.

The proof of Theorem 2 is complete.

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