ORIGINAL PAPER

PARTITIONS OF THE SET OF POSITIVE INTEGERS IN TRANSLATED SETS

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Abstract. *If $F=\{A_i \mid i\in I\}$ is a family of subsets $A_i\subset \mathbf{N}$, $i\in I$, we say that F is a translated family if there is a subset $B\subset\mathbf{N}$ and a sequence $\{a_i \mid i\in I\}\subset\mathbf{Z}$ such that $B=a_i+A_i$, $i\in I$. In this paper we study the existence of partitions of the set $\mathbf{N}$ in translated sets.*

Keywords: *Translated families, partitions, arithmetic progression.*

1. INTRODUCTION

Many of actual problems regarding partition and in particular partitions of **N** and **Z** are from Ramsey Theory: a collection of results which, given a finite partition (coloring) of some structure guarantee the existence of certain monochromatic configuration or substructures. Van der Waerden's Theorem [2] is the result which has determined the research in Ramsey Theory of Integers [1].

In this paper we study a different type of problems: the possibility of partition of **N** in translated sets, problems studied in [3] and [4].

2. INFINITE PARTITIONS IN TRANSLATED SETS

The problems which we discuss in this section is whether there are partitions of the set **N** in an infinity of translated sets, that is there are subsets $B\subset\mathbf{N}$ and sequences $\{a_n \mid n\in\mathbf{N}\}\subset\mathbf{N}$ such that the sets $A_n=a_n+B$, $n\in\mathbf{N}$ form a partition of **N**.

First we notice that any two translated sets have the same cardinality, so the sets A_n, $n\in\mathbf{N}$ are all finite or all infinite. In the case of finite sets we can construct infinite partition of any cardinal $k\in\mathbf{N}$:

Example 2.1. We take $B=\{0,1,\ldots,k-1\}=[1,k-1]$ and $a_n=n\cdot k$, $n\in\mathbf{N}$.

We have: $A_0=0+B=B$, $A_1=k+B=[k,2k-1]$, $A_2=2k+B=[2k,3k-1],\ldots,$ $A_n=nk+B=[nk,(n+1)k-1],\ldots$, and the sets $A_0,A_1,A_2,\ldots,A_n,\ldots$ form a partition of **N**.

It is obvious that using this idea it is not possible to construct partitions in infinite sets but the answers for the case of infinite sets is affirmative.

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Theorem 2.1. *There are partitions of* $\mathbf{N}$ *into an infinite number of infinite translated sets.*

Proof: We look for a partition of the form $\mathbf{N}=\bigcup_{n=0}^{\infty} A_n$, in which all the sets A_n are infinite and can be obtained by the translation of the set $B=A_0$, the set from the partition which contain 0.

Let $T=\{t_0,t_1,t_2,...,t_n,...\}$ be the set of translations and let $A_n=t_n+B$, $n\in\mathbf{N}$. The condition that the sets A_n, $n\in\mathbf{N}$ form a partition of $\mathbf{N}$ is about showing that any number $n\in\mathbf{N}$ can be written uniquely as $n=t+b$, with $t\in T$ and $b\in B$.

We define the set B as the set of natural numbers which in base 2 are of the form: $x=\overline{x_n x_{n-1}...x_2 x_1 x_0}$ and all digits with odd index are 0: $0=x_1=x_3=x_5=...$

We define the set T as the set of numbers x which in base 2 have all the even index digits equal to 0. Note that $\mathbf{N}=T+B$ and that any positive integer n can be written in a unique form: $n=t+b$, $t\in T$ and $b\in B$.

3. FINITE PARTITION OF N IN TRANSLATED SETS

The most famous partition of $\mathbf{N}$ in n subsets is the partition in equivalence classes modulo n: $\mathbf{N}=C_0\cup C_1\cup...\cup C_{n-1}$, where $C_0=n\cdot\mathbf{N}$, $C_1=1+n\cdot\mathbf{N}$,..., $C_{n-1}=n-1+n\cdot\mathbf{N}$, such that the sets $C_1,C_2,...,C_{n-1}$ can be obtained by the translation of the set C_0 and we have the relations:

$$n+C_0=n-1+C_1=n-2+C_2=...=1+C_{n-1}=n\cdot\mathbf{N}^*.$$

We propose to characterize the n-tuples $(a_1,a_2,...,a_n)$ for which there are partition of the set $\mathbf{N}$: $\mathbf{N}=A_1\cup A_2\cup...\cup A_n$ such that $a_1+A_1=a_2+A_2=...=a_n+A_n$.

We obtain a characterization of arithmetic progression with a prime number of terms.

Theorem 3.1. *Let $n\geq 3$ be a prime number and let $a_1<a_2<...<a_n$ be integers.*

Then $a_1,a_2,...,a_n$ is an arithmetic progression if and only if there exists a partition of $\mathbf{N}$ *with classes $A_1,A_2,...,A_n$ such that $a_1+A_1=a_2+A_2=...=a_n+A_n$.*

Proof: If $a_1,a_2,...,a_n$ is an arithmetic progression with step r then the partition $(A,A-r,A-2r,...,A-(n-1)r)$, with $A=\bigcup_{k\geq 0}\{knr+(n-1)r+i \mid i\in\overline{0,r-1}\}$ fulfils the required condition.

For the converse, denote $r_i=a_n-a_{n-i}$ and $B_i=A_{n-i}$ for $i\in\overline{0,n-1}$, so $B_i=B_0+r_i$ for $i\geq 1$. We will call segment of length k of the set B_i every set $\{a,a+1,...,a+k-1\}\subset B_i$ such that $a-1\notin B_i$ and $a+k\notin B_i$.

We will firstly prove that each B_i is an union of segments of the same lengths $r=r_1$.

From $x\in B_0\Rightarrow x+r\in B_1$ we get $x\in B_0\Rightarrow x+r\notin B_0$, therefore all the segments of B_0 must have lengths less that $r+1$. If B_0 contains segments of lengths less than r, let S be the first of these. Then $S+r\subset B_1$ and between S and $S+r$ there exists a segment S' of some B_i, $i\neq 0$. But, in this case the segment $S'-r_i$ would be in B_0, would have length less that r and would be before S, which contradicts the way S was chosen.

This proves that all the segments of B_0 have length r and, from $B_i=B_0+r_i$, this is also true for every B_i.

We will now prove that the first segment of B_i is $S_i=\{ir,ir+1,...,ir+r-1\}$ for every $i\in\overline{0,n-1}$. Since $x\in B_i$ for $i\geq 1\Rightarrow x\geq r_i\geq r$ it follows that B_0 must contain the segment $S_0=\{0,1,...,r-1\}$ and therefore B_1 contains the segment $S_1=\{r,r+1,...,2r-1\}$.

Suppose now that there exists $k<n$ $(k\geq 2)$ such that $S_0\in B_0$, $S_1\in B_1$,..., S_{k-1} and $S_k\notin B_k$. Then S_k must be a segment of some B_i (since all the segments have the same length), i must be less than k and B_i must be B_0 (because the second segment of every B_i, $i\geq 1$ must come after the second segment of B_0).

This leads to $S_{k+1}\in B_0$, $S_{k+2}\in B_2$,..., $S_{2k-1}\in B_{k-1}$ and, repeating the above judgement if necessary, the first segment of B_k must be of the form S_{lk}, $l\geq 2$. This leads to $r_k=lk$, therefore $S_{(l+1)k}=S_k+r_k\in B_k$. The segment S_{lk+1} cannot be in B_0 (it would lead to $S_{(l+1)k}\in B_{k-1}$) or in any of the $B_i's$, $i\in\overline{0,k-1}$ (the set $\{S_0,S_1,...,S_{lk+1}\}$ would contain more segments from B_i than segments from B_0) therefore $S_{lk+1}\in B_{k+1}$.

In the same way $S_{lk+2}\in B_{k+2}$,..., $S_{lk+k-1}\in B_{2k-1}$ and the sequence of segments from $(B_k,B_{k+1},...,B_{2k-1})$ will repeat itself a number of times before the appearance of a segment from a new set (which might be B_0 or B_{2k}).

We notice that a judgement as above shows that each time when a segment from a new set B_{sk} appears, then he must be followed immediately by segments from the sets $B_{sk+1},B_{sk+2},...,B_{sk+k-1}$, so the number n of $B_i's$ must be a multiple of k, $1<k<n$, a contradiction with the premises.

Thus $S_i\in B_i$ for every $i\in\overline{0,n-1}$, therefore $r_i=ir$ for every $i\in\overline{1,n-1}$ and $a_1,a_2,...,a_n$ is an arithmetic progression.

Remark 3.1. If the number of terms of progression is not a prime number one of the implication of Theorem 3.1 is not valid. More precisely, there are partitions of the set $\mathbf{N}$ in $n=p\cdot q$ translated subsets ($p\geq 2$, $q\geq 2$) but the set T of translations does not form an arithmetic progression, as we shall see in the next example.

Example 3.1. $B_0=\{2nk\mid k\in\mathbf{N}\}\cup\{2nk+p\mid k\in\mathbf{N}\}$ and $r_i=i+p\left[\frac{i}{p}\right]$ for $i\in\overline{1,n-1}$ which corresponds to the periodic sequence of segments of length 1 obtained by repeating the block $[B_0B_1...B_{p-1}B_0...B_{p-1}]$ $[B_pB_{p+1}...B_{2p-1}B_p...B_{2p-1}]$... $[B_{n-p}B_{n-p+1}...B_{n-1}B_{n-p}...B_{n-1}]$.

For $p=q=2$, $n=4$, we obtain the partition:

$$A_0=\{8k,8k+2\mid k\in\mathbf{N}\},\ A_1=\{8k+1,8k+3\mid k\in\mathbf{N}\},$$

$$A_2=\{8k+4,8k+6\mid k\in\mathbf{N}\},\ A_3=\{8k+5,8k+7\mid k\in\mathbf{N}\}$$

$a_0 = 5$, $a_1 = 4$, $a_2 = 1$, $a_3 = 0$, which is not an arithmetic progression, but:

$$a_0 + A_0 = a_1 + A_1 = a_2 + A_2 = a_3 + A_3 .$$

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