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SOME TRILATERAL GENERATING FUNCTIONS OF BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE JACOBI POLYNOMIALS

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Abstract. *In this note, we have obtained some novel results on trilateral generating functions for the biorthogonal polynomials suggested by the classical Jacobi polynomials,* $J_n(\alpha,\beta,k;x)$ *with Tchebycheff polynomials by group theoretic method. As special cases, we have obtained the corresponding results on Jacobi polynomials.*

Keywords: *Biorthogonal polynomials, Jacobi polynomials, trilateral generating functions.*

AMS-2010 Subject Classification Code: *33C45, 33C47, 33C65.*

1. INTRODUCTION

In 1982 [3], Madhekar and Thakare studied a pair of biorthogonal polynomials $J_n(\alpha,\beta,k;x)$ and $K_n(\alpha,\beta,k;x)$ that are suggested by the classical Jacobi polynomials, where $J_n(\alpha,\beta,k;x)$ and $K_n(\alpha,\beta,k;x)$ are respectively the polynomials each of degree n in x^k and x respectively. Explicit representations for the two polynomials $J_n(\alpha,\beta,k;x)$ and $K_n(\alpha,\beta,k;x)$ are given by:

$$J_n(\alpha,\beta,k;x) = \frac{(1+\alpha)_{kn}}{n\,!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(1+\alpha+\beta+n)_{kj}}{(1+\alpha)_{kj}}\left(\frac{1-x}{2}\right)^{kj} \tag{1.1}$$

and

$$K_n(\alpha,\beta,k;x) = \sum_{r=0}^{n}\sum_{s=0}^{r}(-1)^{r+s}\binom{r}{s}\frac{(1+\beta)_n}{n\,!r!\,(1+\beta)_{n-r}}\left(\frac{s+\alpha+1}{k}\right)_n\left(\frac{x-1}{2}\right)^r\left(\frac{x+1}{2}\right)^{n-r}, \tag{1.2}$$

where $(a)_n$ is the pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & if\ n=0, \quad a\neq 0 \\ a(a+1)\ldots(a+n-1), & \forall\ n\in\{1,2,3\ldots\}\end{cases}$$

$\alpha > -1, \beta > -1, x$ is real and k is a positive integer. For $k = 1$, both (1.1) and (1.2) yield the classical Jacobi polynomials. In the present paper we are interested only on $J_n(\alpha,\beta,k;x)$.

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In [1], Chongdar and Chatterjea gave a general method based on the theory of one parameter group of continuous transformations, with the help of which any unilateral generating relation involving one special function can be transformed into a trilateral generating relation with Tchebycheff polynomials.

In fact in [1], a unilateral generating relation is converted to bilateral generating relation with the help of one parameter group of continuous transformations and then this bilateral generating relation is converted into a trilateral generating relation with the Tchebycheff polynomial by means of the relation

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right].$$

The aim at presenting this paper is to obtain some novel result on trilateral generating relations for the biorthogonal polynomials $J_n(\alpha,\beta,k;x)$ with Tchebycheff polynomial by utilizing the above mentioned method of Chongdar and Chatterjea.

2. ON BIORTHOGONAL POLYNOMIALS, $J_n(\alpha,\beta,k;x)$

For the biorthogonal polynomials suggested by the classical Jacobi polynomials, $J_n(\alpha,\beta,k;x)$ we notice that if we consider the following linear partial differential operator:

$$R = \frac{(1-x)z}{y}\frac{\partial}{\partial x} - z\frac{\partial}{\partial y}$$

such that

$$R\left(J_n(\alpha,\beta,k;x)\,y^\alpha z^\beta\right) = (-\alpha-kn)\,J_n(\alpha-1,\beta+1,k;x)\,y^{\alpha-1}z^{\beta+1}. \quad (2.1)$$

The extended form of the group generated by R is given by

$$e^{wR}f(x,y,z) = f\left(x+(1-x)\frac{wz}{y},\ y\left(1-\frac{wz}{y}\right),z\right), \quad (2.2)$$

where $f(x,y,z)$ is an arbitrary function, w is an arbitrary constant.

At first, let us consider the following unilateral generating relation:

$$G(x,w) = \sum_{\beta=0}^{\infty} a_\beta\, J_n(\alpha,\beta,k;x)\, w^\beta. \quad (2.3)$$

Replacing w by wvz and then multiplying both sides of (2.3) by y^α, we get

$$y^\alpha\, G(x,wvz) = \sum_{\beta=0}^{\infty} a_\beta\left(J_n(\alpha,\beta,k;x)\,y^\alpha z^\beta\right)(wv)^\beta. \quad (2.4)$$

Operating e^{wR} on both sides of (2.4), we get

$$e^{wR}\left(y^\alpha\, G(x,wvz)\right) = e^{wR}\left(\sum_{\beta=0}^{\infty} a_\beta\left(J_n(\alpha,\beta,k;x)\,y^\alpha z^\beta\right)(wv)^\beta\right). \quad (2.5)$$

Now the left member of (2.5), with the help of (2.2), reduces to

$$\left(1-\frac{wz}{y}\right)^{\alpha} G\left(x+(1-x)\frac{wz}{y}, wvz\right) y^{\alpha}. \qquad (2.6)$$

The right member of (2.5), with the help of (2.1), becomes

$$\sum_{\beta=0}^{\infty}\sum_{p=0}^{\infty} a_{\beta}\frac{w^{p}}{p\,!}(-\alpha-kn)_{p}\, J_{n}(\alpha-p,\beta+p,k;x)\, y^{\alpha-p}z^{\beta+p}(wv)^{\beta}$$

$$=\sum_{\beta=0}^{\infty}\sum_{p=0}^{\beta} a_{\beta-p}\frac{(-\alpha-kn)_{p}}{p\,!}\, J_{n}(\alpha-p,\beta,k;x)\, y^{\alpha-p}(wz)^{\beta}v^{\beta-p}. \qquad (2.7)$$

Now equating (2.6) and (2.7) and then substituting $y = z = 1$, we get

$$(1-w)^{\alpha}G(x+(1-x)w, wv) = \sum_{\beta=0}^{\infty}\sigma_{\beta}(x,v)w^{\beta},$$

where

$$\sigma_{\beta}(x,v) = \sum_{p=0}^{\beta} a_{p}\binom{\beta-p-\alpha-nk-1}{-\alpha-nk-1} J_{n}(\alpha-\beta+p,\beta,k;x)v^{p}. \quad (2.8)$$

Now to convert the above bilateral generating relation into a trilateral generating relation with Tchebycheff polynomial, we notice that

$$\sum_{\beta=0}^{\infty}\sigma_{\beta}(x,v)w^{\beta}T_{\beta}(u)$$

$$=\frac{1}{2}[(1-\rho_1)^{\alpha}G(x+(1-x)\rho_1, v\rho_1)+(1-\rho_2)^{\alpha}G(x+(1-x)\rho_2, v\rho_2)],$$

where $\rho_1 = w\left(u+\sqrt{u^2-1}\right)$ and $\rho_2 = w\left(u-\sqrt{u^2-1}\right)$.

Thus we have the following general theorem.

Theorem 1. If there exists a generating relation of the form

$$G(x,w) = \sum_{\beta=0}^{\infty} a_{\beta}\, J_{n}(\alpha,\beta,k;x)w^{\beta}, \qquad (2.9)$$

then

$$\sum_{\beta=0}^{\infty}\sigma_{\beta}(x,v)w^{\beta}T_{\beta}(u)$$

$$=\frac{1}{2}[(1-\rho_1)^{\alpha}G(x+(1-x)\rho_1, v\rho_1)+(1-\rho_2)^{\alpha}G(x+(1-x)\rho_2, v\rho_2)], \quad (2.10)$$

where

$$\sigma_\beta(x,v) = \sum_{p=0}^{\beta} a_p \binom{\beta - p - \alpha - nk - 1}{-\alpha - nk - 1} J_n(\alpha - \beta + p, \beta, k; x) v^p,$$

which does not seem to have appeared in the earlier works.

Special Case 1. By putting $k = 1$ in Theorem 1, we get the following general theorem on Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$:

Theorem 2. If there exists a generating relation of the form

$$G(x,w) = \sum_{\beta=0}^{\infty} a_\beta P_n^{(\alpha,\beta)}(x) w^\beta, \tag{2.11}$$

then

$$\sum_{\beta=0}^{\infty} \sigma_\beta(x,v) w^\beta T_\beta(u)$$

$$= \frac{1}{2}[(1-\rho_1)^\alpha G(x + (1-x)\rho_1, v\rho_1) + (1-\rho_2)^\alpha G(x + (1-x)\rho_2, v\rho_2)], \tag{2.12}$$

where

$$\sigma_\beta(x,v) = \sum_{p=0}^{\beta} a_p \binom{\beta - p - \alpha - n - 1}{-\alpha - n - 1} P_n^{(\alpha-\beta+p,\beta)}(x) v^p,$$

$$\rho_1 = w\left(u + \sqrt{u^2-1}\right) \text{ and } \rho_2 = w\left(u - \sqrt{u^2-1}\right).$$

Now, we would like to mention that the above result (Theorem 1), obtained by double interpretations to both the parameters α and β of the biorthogonal polynomials suggested by the classical Jacobi polynomials, $J_n(\alpha,\beta,k;x)$ can also be easily derived by defining a simple operator with the suitable interpretation of the parameter β while studying $J_n(\alpha-\beta,\beta,k;x)$ $J_n(\alpha-\beta,\beta,k;x)$, a modification of $J_n(\alpha,\beta,k;x)$.

Now we consider the following linear partial differential operator for the modified biorthogonal polynomials, $J_n(\alpha-\beta,\beta,k;x)$:

$$R = y(1-x)\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} - \alpha y$$

such that

$$R\left(J_n(\alpha-\beta,\beta,k;x)\, y^\beta\right) = (-\alpha - kn + \beta)\, J_n(\alpha-\beta-1,\beta+1,k;x)\, y^{\beta+1}. \tag{2.13}$$

The extended form of the group generated by R is given by

$$e^{wR} f(x,y) = (1-wy)^\alpha f\left(x + w(1-x)y, \frac{y}{1-wy}\right), \tag{2.14}$$

where $f(x,y)$ is an arbitrary function, w is an arbitrary constant.

Now using (2.14), we obtain

$$e^{wR}\left(J_n(\alpha-\beta,\beta,k;x)\, y^\beta\right) = (1-wy)^{\alpha-\beta} J_n(\alpha-\beta,\beta,k;x + w(1-x)y)\, y^\beta. \tag{2.15}$$

But using (2.13), we obtain

$$e^{wR}\left(J_n(\alpha-\beta,\beta,k;x)\, y^\beta\right)$$

$$= \sum_{p=0}^{\infty} \frac{w^p}{p\,!} (-\alpha - kn + \beta)_p \, J_n(\alpha - \beta - p, \beta + p, k; x) \, y^{\beta+p}. \quad (2.16)$$

Now equating (2.15) and (2.16), we get

$$(1 - \mathrm{w}y)^{\alpha-\beta} J_n(\alpha - \beta, \beta, k; \mathrm{x} + (1 - \mathrm{x})\mathrm{w}y)$$
$$= \sum_{p=0}^{\infty} \frac{w^p}{p\,!} (-\alpha - kn + \beta)_p \, J_n(\alpha - \beta - p, \beta + p, k; x) \, y^p. \quad (2.17)$$

Replacing *wy* by *t* in (2.17), we get

$$(1 - \mathrm{t})^{\alpha-\beta} J_n(\alpha - \beta, \beta, k; \mathrm{x} + (1 - \mathrm{x})\mathrm{t})$$
$$= \sum_{p=0}^{\infty} \frac{(-\alpha - kn + \beta)_p}{p\,!} \, J_n(\alpha - \beta - p, \beta + p, k; x) t^p, \quad (2.18)$$

which does not seem to have appeared before.

Finally, if we write $\alpha = \alpha + \beta$ on both sides of (2.18), we get

$$(1 - \mathrm{t})^{\alpha} J_n(\alpha, \beta, k; \mathrm{x} + (1 - \mathrm{x})\mathrm{t}) = \sum_{p=0}^{\infty} \frac{(-\alpha - kn)_p}{p\,!} \, J_n(\alpha - p, \beta + p, k; x) t^p, \quad (2.19)$$

which does not seem to have appeared before.

Corollary 1: For $k = 1$, the generating relation (2.18) reduces to the generating relation involving Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$:

$$(1 - \mathrm{t})^{\alpha-\beta} P_n^{(\alpha-\beta,\beta)}(\mathrm{x} + (1 - \mathrm{x})\mathrm{t}) = \sum_{p=0}^{\infty} \frac{(-\alpha - n + \beta)_p}{p\,!} \, P_n^{(\alpha-\beta-p,\beta+p)}(x) t^p, \quad (2.20)$$

which is found derived in [4].

Corollary 2: If we write $\alpha = \alpha + \beta$ on both sides of (2.20), we get

$$(1 - \mathrm{t})^{\alpha} P_n^{(\alpha,\beta)}(\mathrm{x} + (1 - \mathrm{x})\mathrm{t}) = \sum_{p=0}^{\infty} \frac{(-\alpha - n)_p}{p\,!} \, P_n^{(\alpha-p,\beta+p)}(x) t^p, \quad (2.21)$$

which is found derived in [2,4].

We now proceed to prove the Theorem 1 by using (2.19).

Proof of Theorem 1:

$$\sum_{\beta=0}^{\infty} \sigma_\beta(x,v) w^\beta T_\beta(u)$$
$$= \sum_{\beta=0}^{\infty} \sigma_\beta(x,v) w^\beta \frac{1}{2}\left[\left(u + \sqrt{u^2 - 1}\right)^\beta + \left(u - \sqrt{u^2 - 1}\right)^\beta\right]$$
$$= \frac{1}{2} \sum_{\beta=0}^{\infty} \sigma_\beta(x,v) w^\beta \left(u + \sqrt{u^2 - 1}\right)^\beta + \frac{1}{2} \sum_{\beta=0}^{\infty} \sigma_\beta(x,v) w^\beta \left(u - \sqrt{u^2 - 1}\right)^\beta$$

$$=\frac{1}{2}\sum_{\beta=0}^{\infty}\sum_{p=0}^{\beta} a_p \binom{\beta-p-\alpha-kn-1}{-\alpha-kn-1} J_n(\alpha-\beta+p,\beta,k;x) v^p w^\beta \left(u+\sqrt{u^2-1}\right)^\beta$$

$$+\frac{1}{2}\sum_{\beta=0}^{\infty}\sum_{p=0}^{\beta} a_p \binom{\beta-p-\alpha-kn-1}{-\alpha-kn-1} J_n(\alpha-\beta+p,\beta,k;x) v^p w^\beta \left(u-\sqrt{u^2-1}\right)^\beta$$

$$=\frac{1}{2}\sum_{p=0}^{\infty} a_p (v\rho_1)^p \sum_{\beta=0}^{\infty}\frac{(-\alpha-kn)_\beta}{\beta\,!} J_n(\alpha-\beta,\beta+p,k;x)\,(\rho_1)^\beta$$

$$+\frac{1}{2}\sum_{p=0}^{\infty} a_p (v\rho_2)^p \sum_{\beta=0}^{\infty}\frac{(-\alpha-kn)_\beta}{\beta\,!} J_n(\alpha-\beta,\beta+p,k;x)\,(\rho_2)^\beta$$

$$=\frac{1}{2}(1-\rho_1)^\alpha \sum_{p=0}^{\infty} a_p J_n(\alpha,p,k;\mathrm{x}+(1-\mathrm{x})\rho_1)(v\rho_1)^p$$

$$+\frac{1}{2}(1-\rho_2)^\alpha \sum_{p=0}^{\infty} a_p J_n(\alpha,p,k;\mathrm{x}+(1-\mathrm{x})\rho_2)(v\rho_2)^p$$

$$=\frac{1}{2}\left[(1-\rho_1)^\alpha G(\mathrm{x}+(1-\mathrm{x})\rho_1, v\rho_1)+(1-\rho_2)^\alpha G(\mathrm{x}+(1-\mathrm{x})\rho_2, v\rho_2)\right],$$

where

$$\rho_1 = w\left(u+\sqrt{u^2-1}\right) \text{ and } \rho_2 = w\left(u-\sqrt{u^2-1}\right),$$

which is Theorem 1.

Furthermore, we would like to point it out that the following theorem on bilateral generating function involving $J_n(\alpha\text{-}\beta,\beta,k;x)$ is the direct consequence of our result (2.18).

Theorem 3. If there exists a generating relation of the form

$$G(x,w) = \sum_{\beta=0}^{\infty} a_\beta J_n(\alpha-\beta,\beta,k;x) w^\beta, \tag{2.22}$$

then

$$(1-\mathrm{t})^\alpha G\left(\mathrm{x}+\mathrm{w}(1-\mathrm{x})\mathrm{t}, \frac{vt}{1-\mathrm{t}}\right) = \sum_{\beta=0}^{\infty} \sigma_\beta(v) J_n(\alpha-\beta,\beta,k;x) t^\beta, \tag{2.23}$$

where

$$\sigma_\beta(v) = \sum_{p=0}^{\beta} a_p \binom{\beta-\alpha-nk-1}{p-\alpha-nk-1} v^p.$$

Proof of Theorem 3:

$$\sum_{\beta=0}^{\infty} \sigma_\beta(v) J_n(\alpha-\beta,\beta,k;x)\, t^\beta$$

$$=\sum_{\beta=0}^{\infty}\sum_{p=0}^{\beta} a_p \binom{\beta-\alpha-nk-1}{p-\alpha-nk-1} J_n(\alpha-\beta,\beta,k;x)\, t^\beta v^p$$

$$= \sum_{p=0}^{\infty} a_p (vt)^p \sum_{\beta=0}^{\infty} \frac{(-\alpha - kn + p)_\beta}{\beta\,!} \; J_n(\alpha - \beta - p, \beta + p, k; x) \; (t)^\beta$$

$$= (1-t)^\alpha \sum_{p=0}^{\infty} a_p \, J_n(\alpha - p, p, k; x + (1-x)t) \left(\frac{vt}{1-t}\right)^p$$

$$= (1-t)^\alpha G\left(x + w(1-x)t, \frac{vt}{1-t}\right),$$

which is Theorem 3.

Finally, we shall convert the above bilateral generating relation into a trilateral generating relation with Tchebycheff polynomial and obtain the following theorem:

Theorem 4. If there exists a generating relation of the form

$$G(x,w) = \sum_{\beta=0}^{\infty} a_\beta \, J_n(\alpha - \beta, \beta, k; x) w^\beta, \tag{2.24}$$

then

$$\sum_{\beta=0}^{\infty} \sigma_\beta(v) \, J_n(\alpha - \beta, \beta, k; x) t^\beta \, T_\beta(u)$$

$$= \frac{1}{2}\left[(1-\rho_1)^\alpha G\left(x + (1-x)\rho_1, \frac{v\rho_1}{1-\rho_1}\right) + (1-\rho_2)^\alpha G\left(x + (1-x)\rho_2, \frac{v\rho_2}{1-\rho_2}\right)\right], \tag{2.25}$$

where

$$\sigma_\beta(v) = \sum_{p=0}^{\beta} a_p \binom{\beta - \alpha - nk - 1}{p - \alpha - nk - 1} v^p,$$

$$\rho_1 = t\left(u + \sqrt{u^2 - 1}\right) \text{ and } \quad \rho_2 = t\left(u - \sqrt{u^2 - 1}\right),$$

which does not seem to have appeared in the earlier works.

Special Case 2. By putting $k = 1$ in Theorem 4, we get the following general theorem on Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$:

Theorem 5: If there exists a generating relation of the form

$$G(x,w) = \sum_{\beta=0}^{\infty} a_\beta P_n^{(\alpha-\beta,\beta)}(x) w^\beta, \tag{2.26}$$

then

$$\sum_{\beta=0}^{\infty} \sigma_\beta(v) P_n^{(\alpha-\beta,\beta)}(x) t^\beta \, T_\beta(u)$$

$$= \frac{1}{2}\left[(1-\rho_1)^\alpha G\left(x + (1-x)\rho_1, \frac{v\rho_1}{1-\rho_1}\right) + (1-\rho_2)^\alpha G\left(x + (1-x)\rho_2, \frac{v\rho_2}{1-\rho_2}\right)\right], \tag{2.27}$$

where

$$\sigma_\beta(v) = \sum_{p=0}^{\beta} a_p \binom{\beta-\alpha-n-1}{p-\alpha-n-1} v^p,$$

$$\rho_1 = t\left(u+\sqrt{u^2-1}\right) \quad \text{and} \quad \rho_2 = t\left(u-\sqrt{u^2-1}\right),$$

which does not seem to have appeared in the earlier works.

3. CONCLUSIONS

From the above discussion, it is clear that whenever one knows a unilateral generating relation of the form (2.9, 2.11, 2.24, 2.26) then the corresponding trilateral generating relation can at once be written down from (2.10, 2.12, 2.25, 2.27). So one can get a large number of trilateral generating relations by attributing different suitable values to α_β in (2.9, 2.11, 2.24, 2.26).

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